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ASYMPTOTIC ANALYSIS OF A MICROPOLAR FLUID FLOW IN THIN DOMAIN WITH A FREE AND ROUGH BOUNDARY

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Abstract. Motivated by lubrication problems, we consider a micropolar fluid flow in a 2D domain with a rough and free boundary. We assume that the thickness and the roughness are both of order $0 < \varepsilon << 1$. We prove the existence and uniqueness of a solution of this problem for any value of ε and we establish some a priori estimates. Then we use the two-scale convergence technique to derive the limit problem when ε tends to zero. Moreover we show that the limit velocity and micro-rotation fields are uniquely determined via auxiliary well-posed problems and the limit pressure is given as the unique solution of a Reynolds equation.

Key words. Lubrication, micropolar fluid, free and rough boundary, asymptotic analysis, two-scale convergence, Reynolds equation.

AMS subject classifications. 35Q35, 76A05, 76D08, 76M50.

1. Introduction.

The theory of micropolar fluids, was introduced and formulated by A.C. Eringen in [13]. It aims to describe fluids containing suspensions of rigid particles in a viscous medium. Such fluids exhibit micro-rotational effects and micro-rotational inertia. Therefore they can support couple stress and distributed body couples. They form a class of fluids with nonsymmetric stress tensor for which the classical Navier-Stokes theory is inadequate since it does not take into account the effects of the micro-rotation. Experimental studies have showned that the micropolar model better represents the behavior of numerous fluids such as polymeric fluids, liquid crystals, paints, animal blood, colloidal fluids, ferro-liquids, etc., especially when the characteristic dimension of the flow becomes small (see for instance [26]). Extensive reviews of the theory and its applications can be found in [2, 3] or in the books [14] and [22] and also in more recent articles (see for example [4, 9, 19, 20]).

Motivated by lubrication theory where the domain of flow is usually very thin and the roughness of the boundary strongly affects the flow ([10]), we consider the motion of the micropolar fluid described by the equilibrium of momentum, mass and moment of momentum. More precisely, the velocity field of the fluid $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$, the pressure p^ε and the angular velocity of the micro-rotations of the particles ω^ε satisfy the system

$$u_t^\varepsilon - (\nu + \nu_r)\Delta u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = 2\nu_r \operatorname{rot} \omega^\varepsilon + f^\varepsilon, \tag{1.1}$$

$$\operatorname{div} u^\varepsilon = 0, \tag{1.2}$$

$$\omega_t^\varepsilon - \alpha\Delta\omega^\varepsilon + (u^\varepsilon \cdot \nabla)\,\omega^\varepsilon + 4\nu_r\omega^\varepsilon = 2\nu_r \operatorname{rot} u^\varepsilon + g^\varepsilon, \tag{1.3}$$

in the space-time domain $(0, T) \times \Omega^\varepsilon$ with

$$\Omega^\varepsilon = \{z = (z_1, z_2) \in \mathbb{R}^2, \quad 0 < z_1 < L, \quad 0 < z_2 < \varepsilon h^\varepsilon(z_1)\}, \quad h^\varepsilon(z_1) = h(z_1, \frac{z_1}{\varepsilon})$$

where h is a given smooth function, f^ε and g^ε are given external forces and moments, ν is the usual Newtonian viscosity, ν_r and α are the micro-rotation viscosities, which are assumed to be positive constants ([13]).

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The choice of the domain Ω^ε comes from one of the important fields of the theory of lubrication given by the study of self-lubricating bearings. These bearings are widely used in mechanical and electromechanical industry, to lubricate the main axis of rotation of a device, in order to prevent its endomagement.

Such bearings consist in an inner cylinder and a outer cylinder, and along a circumferencial section, one can see two non-concentric discs. The radii of the two cylinders are much smaller than their lengh and the gap between the two cylinders, which is fullfilled with a lubricant, is much smaller than their radii ([11]). By assuming that the external fields and the flow do not depend on the coordinate along the longitudinal axis of the bearing, one can represent the fluid domain by Ω^ε which is a 2D view of a cross section after a radial cut of the two circumferences. The boundary of Ω^ε is $\partial\Omega^\varepsilon = \bar{\Gamma}_0 \cup \bar{\Gamma}^\varepsilon_L \cup \bar{\Gamma}^\varepsilon_1$,

where $\Gamma_0 = \{z \in \partial\Omega^\varepsilon : z_2 = 0\}$ is the bottom, $\Gamma^\varepsilon_1 = \{z \in \partial\Omega^\varepsilon : z_2 = \varepsilon h^\varepsilon(z_1)\}$ is the upper strongly oscillating part, and Γ^ε_L is the lateral part of the boundary. The surface of the inner cylinder, which corresponds to Γ_0, is in contact with the rotating axis of the device while the surface of the outer cylinder, which corresponds to Γ^ε_1, remains still.

Hence the boundary and initial conditions are given as follows

$$\omega^\varepsilon, \quad u^\varepsilon, \quad p^\varepsilon \quad \text{are L-periodic with respect to } z_1 \tag{1.4}$$

$$u^\varepsilon = U_0 e_1 = (U_0, 0), \quad \omega^\varepsilon = W_0 \quad \text{on} \quad (0,T) \times \Gamma_0 \tag{1.5}$$

$$\omega^\varepsilon = 0, \quad u^\varepsilon \cdot n = 0, \qquad \frac{\partial u^\varepsilon}{\partial n} \cdot \tau = 0 \qquad \text{on} \quad (0,T) \times \Gamma^\varepsilon_1 \tag{1.6}$$

$$u^\varepsilon(0,z) = u^\varepsilon_0(z), \quad \omega^\varepsilon(0,z) = \omega^\varepsilon_0(z) \quad \text{for} \quad z \in \Omega^\varepsilon \tag{1.7}$$

where τ and n are respectively the tangent and normal unit vectors to the boundary of the domain Ω^ε. Let us observe that (1.5) represents non-homogeneous Dirichlet conditions along Γ_0, which means adherence of the fluid to the boundary of the rotating inner cylinder, so U_0 and W_0 are two given functions of the time variable only. The second condition in (1.6) is the nonpenetration boundary condition, while the last one is non-standard, and it means that the tangential component of the flux on Γ^ε_1 is equal zero ([12]).

The choice of the particular scaling, with a roughness in inverse proportion to the thickness of the domain, is quite classical in lubrication theory. In [8] and in [10] a Stokes flow is considered with adhering boundary conditions and Tresca boundary conditions at the fluid solid interface respectively. For other related works see also [6, 7] or [5] for instance.

We prove the existence and uniqueness of a weak solution $(u^\varepsilon, \omega^\varepsilon, p^\varepsilon)$ in adequate functional framework. Then we will establish some a priori estimates for the velocity, micro-rotation and pressure fields, independently of ε, and finally we will derive and study the limit problem when ε tends to zero.

The paper is organized as follows. In Section 2 we give the variational formulation. Then, using an idea of J.L. Lions ([21]), we consider the divergence free condition (1.2) as a constraint, which can be penalized, and we prove in Theorem 2.2 the existence and uniqueness of a weak solution $(u^\varepsilon, \omega^\varepsilon, p^\varepsilon)$ for any value of ε. Let us emphasize that our proof ensures that the pressure (unique up to an additional function of time) belong to $H^{-1}(0,T,L^2_0(\Omega^\varepsilon))$. This result is more suitable for the next parts of our study, than $W^{-1,\infty}(0,T,L^2_0(\Omega^\varepsilon))$ obtained by J. Simon [27] (see also Theorem 2.1 in [16]).

In Section 3, we establish some a priori estimates for the velocity and micro-rotation fields in Proposition 3.2 and for the pressure in Proposition 3.3. In Section 4, since we deal with an evolution problem, we extend first the classical two-scale convergence results ([1, 25]) to a time-dependent setting and we use this technique to prove some convergence properties for the velocity in Proposition 4.3, the micro-rotation in Proposition 4.4, and the pressure in Proposition 4.5.

Then, in Section 5 we derive the limit problem when ε tends to zero in Theorem 5.1. We notice that the trilinear and rotational terms, as well as the time derivative do not contribute when we pass to the limit. However the time variable remains in the limit problem as a parameter. We note also that the limit problem can be easily decoupled: we obtain a variational equality involving only the limit velocity and the limit pressure and another variational equality involving the limit micro-rotation. However, the micropolar nature of the fluid still appears in the limit problem for the velocity and pressure since we keep the viscosity $\nu + \nu_r$. Moreover we show in Proposition 5.2 that the limit velocity and micro-rotation fields are uniquely determined via auxiliary well-posed problems. In Proposition 5.3, we prove that the limit pressure is given as the unique solution of a Reynolds equation. Finally in Section 6 we propose a generalization to the case where both the upper and the lower boundary of the fluid domain are oscillating.

2. Existence and uniqueness results.

We assume that

$$\frac{L}{\varepsilon} \in \mathbb{N}, \quad h : (z_1, \eta_1) \mapsto h(z_1, \eta_1) \text{ is L-periodic in } z_1 \text{ and 1-periodic in } \eta_1, \tag{2.1}$$

so h is L-periodic in z_1. We assume also that

$$h \in \mathcal{C}^\infty([0, L] \times \mathbb{R}), \quad \frac{\partial h}{\partial \eta_1} \text{ is 1-periodic in } \eta_1, \tag{2.2}$$

and there exist h_m and h_M such that

$$0 < h_m = \min_{[0,L]\times[0,1]} h(z_1, \eta_1), \quad \text{and} \quad h_M = \max_{[0,L]\times[0,1]} h(z_1, \eta_1). \tag{2.3}$$

Lemma 2.1. *Let the functions $\mathcal{U}$, $\mathcal{W}$ be in $\mathcal{D}(-\infty, h_m)$, and U_0, W_0 be in $H^1(0,T)$, with $\mathcal{U}(0) = 1$, $\mathcal{W}(0) = 1$. We set*

$$U^\varepsilon(t, z_2) = \mathcal{U}^\varepsilon(z_2) U_0(t) = \mathcal{U}(\frac{z_2}{\varepsilon}) U_0(t), \qquad W^\varepsilon(t, z_2) = \mathcal{W}^\varepsilon(z_2) W_0(t) = \mathcal{W}(\frac{z_2}{\varepsilon}) W_0(t).$$

Then we have for all $(t, z_1) \in (0,T) \times (0, L)$

$$U^\varepsilon(t, 0) = U_0(t), \; U^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = 0, \; \frac{\partial U^\varepsilon}{\partial z_2}(t, \varepsilon h^\varepsilon(z_1)) = 0, \tag{2.4}$$

$$W^\varepsilon(t, 0) = W_0(t), \quad W^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = 0. \tag{2.5}$$

Proof. Indeed, $U^\varepsilon(t,0) = \mathcal{U}(0)U_0(t) = U_0(t)$, $U^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = \mathcal{U}(h(z_1, \frac{z_1}{\varepsilon}))U_0(t) = 0$ and

$$\frac{\partial U^\varepsilon}{\partial z_2}(t, \varepsilon h^\varepsilon(z_1)) = \frac{1}{\varepsilon} \mathcal{U}'(h^\varepsilon(z_1)) U_0(t) = \frac{1}{\varepsilon} \mathcal{U}'(h(z_1, \frac{z_1}{\varepsilon})) U_0(t) = 0,$$

thus (2.4) follows. The proof is valid also for (2.5). ◻

We can now set

$$u^\varepsilon(t,z_1,z_2) = U^\varepsilon(t,z_2)e_1 + v^\varepsilon(t,z_1,z_2) \tag{2.6}$$

$$\omega^\varepsilon(t,z_1,z_2) = W^\varepsilon(t,z_2) + Z^\varepsilon(t,z_1,z_2) \tag{2.7}$$

with U^ε and W^ε satisfying (2.4) (2.5). Moreover

$$\frac{\partial u_i^\varepsilon}{\partial z_j} = \frac{\partial v_i^\varepsilon}{\partial z_j} + \frac{\partial}{\partial z_j}(U^\varepsilon(\cdot,z_2)e_1) = \begin{cases} \frac{\partial v_i^\varepsilon}{\partial z_j} & \text{if} \quad j=1, \\ \frac{\partial v_i^\varepsilon}{\partial z_j} + \frac{\partial U^\varepsilon}{\partial z_2}(\cdot,z_2)e_1 & \text{if} \quad j=2 \end{cases}$$

and from (2.4) $\frac{\partial U^\varepsilon}{\partial z_2}(t,z_2) = \frac{\partial U^\varepsilon}{\partial z_2}(t,\varepsilon h^\varepsilon(z_1)) = 0$ for $(t,z_2)\in(0,T)\times\Gamma_1^\varepsilon$ so

$$\frac{\partial u_i^\varepsilon}{\partial z_j} = \frac{\partial v_i^\varepsilon}{\partial z_j} \quad \text{for} \quad j=1,2 \quad \text{on} \quad (0,T)\times\Gamma_1^\varepsilon. \tag{2.8}$$

Recall also that

$$\operatorname{rot} u^\varepsilon = \frac{\partial u_2^\varepsilon}{\partial z_1} - \frac{\partial u_1^\varepsilon}{\partial z_2}, \qquad \operatorname{rot}\,\omega^\varepsilon = \left(\frac{\partial \omega^\varepsilon}{\partial z_2}, -\frac{\partial \omega^\varepsilon}{\partial z_1}\right).$$

Then the problem (1.1)-(1.7) becomes

$$\begin{aligned} v_t^\varepsilon - (\nu+\nu_r)\Delta v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + U^\varepsilon\frac{\partial v^\varepsilon}{\partial z_1} + (v^\varepsilon)_2\frac{\partial U^\varepsilon}{\partial z_2}e_1 + \nabla p^\varepsilon = 2\nu_r \operatorname{rot} Z^\varepsilon \\ +(\nu+\nu_r)\frac{\partial^2 U^\varepsilon}{\partial z_2^2}e_1 + 2\nu_r\frac{\partial W^\varepsilon}{\partial z_2}e_1 - \frac{\partial U^\varepsilon}{\partial t}e_1 + f^\varepsilon \quad \text{in } (0,T)\times\Omega^\varepsilon, \end{aligned} \tag{2.9}$$

$$\operatorname{div} v^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \quad t\in(0,T), \tag{2.10}$$

$$\begin{aligned} Z_t^\varepsilon - \alpha\Delta Z^\varepsilon + (v^\varepsilon\cdot\nabla)Z^\varepsilon + 4\nu_r Z^\varepsilon + U^\varepsilon\frac{\partial Z^\varepsilon}{\partial z_1} + (v^\varepsilon)_2\frac{\partial W^\varepsilon}{\partial z_2} = 2\nu_r\operatorname{rot} v^\varepsilon \\ +\alpha\frac{\partial^2 W^\varepsilon}{\partial z_2^2} - 2\nu_r\frac{\partial U^\varepsilon}{\partial z_2} - 4\nu_r W^\varepsilon - \frac{\partial W^\varepsilon}{\partial t} + g^\varepsilon \quad \text{in } (0,T)\times\Omega^\varepsilon, \end{aligned} \tag{2.11}$$

$$v^\varepsilon, Z^\varepsilon \text{ and } p^\varepsilon \text{ L-periodic in } z_1, \tag{2.12}$$

$$Z^\varepsilon = 0, \quad v^\varepsilon = 0 \quad \text{on} \quad (0,T)\times\Gamma_0, \tag{2.13}$$

$$Z^\varepsilon = 0, \quad v^\varepsilon\cdot n = 0, \qquad \frac{\partial v^\varepsilon}{\partial n}\cdot\tau = 0 \qquad \text{on} \quad (0,T)\times\Gamma_1^\varepsilon, \tag{2.14}$$

$$v^\varepsilon(0,z) = v_0^\varepsilon(z) = u_0^\varepsilon(z) - U^\varepsilon(0,z_2)e_1 \quad \text{in } \Omega^\varepsilon, \tag{2.15}$$

$$Z^\varepsilon(0,z) = Z_0^\varepsilon(z) = \omega_0^\varepsilon(z) - W^\varepsilon(0,z_2) \quad \text{in } \Omega^\varepsilon, \tag{2.16}$$

where we have denoted by $(v^\varepsilon)_2$ the second component of v^ε.

To define the weak formulation of the above problem (2.9)- (2.16), we recall that Γ_1^ε is defined by the equation $z_2 = \varepsilon h^\varepsilon(z_1)$, thus the unit outward normal vector to Γ_1^ε is given by

$$n = \frac{1}{\sqrt{1 + (\varepsilon (h^\varepsilon)'(z_1))^2}} (-\varepsilon (h^\varepsilon)'(z_1), 1)$$

and $v \cdot n = 0$ becomes $-\varepsilon (h^\varepsilon)'(z_1) v_1 + v_2 = 0$ on Γ_1^ε. We consider now the following functional framework

$$\tilde{V^\varepsilon} = \{ v \in \mathcal{C}^\infty(\overline{\Omega^\varepsilon})^2 : \ v \text{ is L-periodic in } \ z_1, \ v_{|\Gamma_0} = 0, \ -\varepsilon (h^\varepsilon)'(z_1) v_1 + v_2 = 0 \text{ on } \Gamma_1^\varepsilon \}$$

$$\tilde{H^1}^\varepsilon = \{ Z \in \mathcal{C}^\infty(\overline{\Omega^\varepsilon}) : \ Z \text{ is L-periodic in } \ z_1, \ Z = 0 \text{ on } \Gamma_0 \cup \Gamma_1^\varepsilon \}$$

$$V^\varepsilon = \text{closure of } \tilde{V^\varepsilon} \text{ in } H^1(\Omega^\varepsilon) \times H^1(\Omega^\varepsilon), \qquad V^\varepsilon_{div} = \{ v \in V^\varepsilon \ : \ \operatorname{div} v = 0, \text{ in } \Omega^\varepsilon \}$$

$$H^\varepsilon = \text{closure of } \tilde{V^\varepsilon} \text{ in } L^2(\Omega^\varepsilon) \times L^2(\Omega^\varepsilon), \qquad H^{1,\varepsilon} = \text{closure of } \tilde{H^1}^\varepsilon \text{ in } H^1(\Omega^\varepsilon),$$

$$H^{0,\varepsilon} = \text{closure of } \tilde{H^1}^\varepsilon \text{ in } L^2(\Omega^\varepsilon), \quad L^2_0(\Omega^\varepsilon) = \{ q \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} q(z) dz = 0 \}.$$

We endowed these functional spaces with the inner products and norms defined by

$$[\bar{v}, \Theta] = (v, \varphi) + (Z, \psi) \text{ in } H^\varepsilon \times H^{0,\varepsilon} \text{ with the norm } [\bar{v}] = [\bar{v}, \bar{v}]^{\frac{1}{2}}$$

$$[[\bar{v}, \Theta]] = (\nabla v, \nabla \varphi) + (\nabla Z, \nabla \psi) \text{ in } V^\varepsilon \times H^{1,\varepsilon} \text{ with the norm } [[\bar{v}]] = [[\bar{v}, \bar{v}]]^{\frac{1}{2}}$$

for any pairs of functions $\bar{v} = (v, Z)$ and $\Theta = (\varphi, \psi)$. The weak formulation of the problem (2.9)- (2.16) is given by

Problem (P^ε) Find

$$\bar{v}^\varepsilon = (v^\varepsilon, Z^\varepsilon) \in \Big(\mathcal{C}([0,T]; H^\varepsilon) \cap L^2(0,T; V^\varepsilon_{div}) \Big) \times \Big(\mathcal{C}([0,T]; H^{0,\varepsilon}) \cap L^2(0,T; H^{1,\varepsilon}) \Big)$$

and $p^\varepsilon \in H^{-1}(0,T; L^2_0(\Omega^\varepsilon))$, such that

$$\begin{aligned} &[\frac{\partial \bar{v}^\varepsilon}{\partial t}(t), \Theta^\varepsilon] + a(\bar{v}^\varepsilon(t), \Theta^\varepsilon) + B(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta^\varepsilon) + \mathcal{R}(\bar{v}^\varepsilon(t), \Theta^\varepsilon) = \\ &= (p^\varepsilon(t) \, , \, \operatorname{div} \varphi^\varepsilon) + (\mathcal{F}(\bar{v}^\varepsilon(t)), \Theta^\varepsilon) \qquad \forall \Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon}, \end{aligned} \tag{2.17}$$

with the initial condition

$$\bar{v}^\varepsilon(z, 0) = \bar{v_0^\varepsilon}(z) = (v_0^\varepsilon(z) \, , \, Z_0^\varepsilon(z)), \tag{2.18}$$

where

$$\begin{aligned} (\mathcal{F}(\bar{v}^\varepsilon(t)), \Theta^\varepsilon) &= -a(\bar{\xi^\varepsilon}(t), \Theta^\varepsilon) - B(\bar{\xi^\varepsilon}(t), \bar{v}^\varepsilon(t), \Theta^\varepsilon) - B(\bar{v}^\varepsilon(t), \bar{\xi^\varepsilon}(t), \Theta^\varepsilon) \\ &- \mathcal{R}(\bar{\xi^\varepsilon}(t), \Theta^\varepsilon) - [\frac{\partial \bar{\xi^\varepsilon}}{\partial t}(t), \Theta^\varepsilon] + [\bar{f^\varepsilon}(t), \Theta^\varepsilon], \quad \bar{\xi^\varepsilon} = (U^\varepsilon e_1, W^\varepsilon), \end{aligned} \tag{2.19}$$

and for all $\bar v = (v, Z)$, $\bar u = (u, w)$, and $\Theta = (\varphi, \psi)$ in $V^\varepsilon \times H^{1,\varepsilon}$,

$$\begin{aligned}
\left[\bar f^\varepsilon, \Theta\right] &= (f^\varepsilon, \varphi) + (g^\varepsilon, \psi),\\
a(\bar v, \Theta) &= (\nu + \nu_r)(\nabla v, \nabla \varphi) + \alpha(\nabla Z, \nabla \psi),\\
\mathcal{R}(\bar v, \Theta) &= -2\nu_r(\operatorname{rot} Z, \varphi) - 2\nu_r(\operatorname{rot} v, \psi) + 4\nu_r(Z, \psi),\\
B(\bar v, \bar u, \Theta) &= b(v, u, \varphi) + b_1(v, w, \psi) = \sum_{i,j=1}^{2} \int_{\Omega^\varepsilon} v_i \frac{\partial u_j}{\partial z_i} \varphi_j dz + \sum_{i=1}^{2} \int_{\Omega^\varepsilon} v_i \frac{\partial w}{\partial z_i} \psi dz.
\end{aligned}$$

Theorem 2.2. *Let $T > 0$, U^ε and W^ε be given as in* Lemma 2.1, *f^ε in $(L^2((0,T) \times \Omega^\varepsilon))^2$, g^ε in $L^2((0,T) \times \Omega^\varepsilon)$ and $(v_0^\varepsilon, Z_0^\varepsilon)$ in $H^\varepsilon \times H^{0,\varepsilon}$. Then problem (P^ε) admits a unique solution $(v^\varepsilon, Z^\varepsilon, p^\varepsilon)$.*

Proof. Following the techniques proposed by J.L.Lions in [21], we construct a sequence of approximate solutions by relaxing the divergence free condition for the velocity field. More precisely we consider the following penalized problems (P_δ^ε), with $\delta > 0$:

Problem (P_δ^ε) Find

$$\bar v_\delta^\varepsilon = (v_\delta^\varepsilon, Z_\delta^\varepsilon) \in \Big(\mathcal{C}([0,T]; H^\varepsilon) \cap L^2(0,T; V^\varepsilon)\Big) \times \Big(\mathcal{C}([0,T]; H^{0,\varepsilon}) \cap L^2(0,T; H^{1,\varepsilon})\Big)$$

such that

$$\begin{aligned}
&[\frac{\partial \bar v_\delta^\varepsilon}{\partial t}, \Theta^\varepsilon] + a(\bar v_\delta^\varepsilon, \Theta^\varepsilon) + B(\bar v_\delta^\varepsilon, \bar v_\delta^\varepsilon, \Theta^\varepsilon) + \frac{1}{2}\left\{(v_\delta^\varepsilon \operatorname{div} v_\delta^\varepsilon, \varphi^\varepsilon) + (Z_\delta^\varepsilon \operatorname{div} v_\delta^\varepsilon, \psi^\varepsilon)\right\}\\
&+\frac{1}{\delta}(\operatorname{div} v_\delta^\varepsilon, \operatorname{div} \varphi^\varepsilon) = (\mathcal{F}(\bar v_\delta^\varepsilon), \Theta^\varepsilon) - \mathcal{R}(\bar v_\delta^\varepsilon, \Theta^\varepsilon) \quad \forall \Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon},
\end{aligned} \tag{2.20}$$

with the initial condition

$$\bar v_\delta^\varepsilon(0) = \bar{v_0^\varepsilon}. \tag{2.21}$$

The first term on the right of the second line of (2.20) is the penalty term and the term

$$\frac{1}{2}\left\{(v_\delta^\varepsilon \operatorname{div} v_\delta^\varepsilon, \varphi^\varepsilon) + (Z_\delta^\varepsilon \operatorname{div} v_\delta^\varepsilon, \psi^\varepsilon)\right\}$$

is added in order to vanish with $B(\bar v_\delta^\varepsilon, \bar v_\delta^\varepsilon, \Theta^\varepsilon)$ when $\Theta^\varepsilon = \bar v_\delta^\varepsilon$.

Hence the proof of Theorem 2.2 is divided in two parts. First we prove the existence of a solution of (P_δ^ε), for any $\delta > 0$, by using a Galerkin method. Then we pass to the limit as δ tends to zero by applying compactness arguments and we prove that the limit solves problem (P^ε).

Since V^ε and $H^{1,\varepsilon}$ are closed subspaces of $(H^1(\Omega^\varepsilon))^2$ and $H^1(\Omega^\varepsilon)$, they admit Hilbertian bases, denoted as $(\Phi_j)_{j\geq 1}$ and $(\psi_j)_{j\geq 1}$ respectively, which are orthonormal in $(H^1(\Omega^\varepsilon))^2$ and $H^1(\Omega^\varepsilon)$ and are also orthogonal bases of $(L^2(\Omega^\varepsilon))^2$ and $L^2(\Omega^\varepsilon)$. For all $m \geq 1$ we define v_{0m}^ε and Z_{0m}^ε as the L^2-orthogonal projection of v_0^ε and Z_0^ε on the finite dimentional subspaces $\langle \Phi_1, \ldots, \Phi_m \rangle$ and $\langle \psi_1, \ldots, \psi_m \rangle$ respectively and we let $\bar v_{0m}^\varepsilon = (v_{0m}^\varepsilon, Z_{0m}^\varepsilon)$. Then we consider $\bar v_{\delta m}^\varepsilon = (v_{\delta m}^\varepsilon, Z_{\delta m}^\varepsilon)$, with

$$v_{\delta m}^\varepsilon(t,x) = \sum_{j=1}^{m} v_{\delta m j}^\varepsilon(t) \Phi_j(x), \qquad Z_{\delta m}^\varepsilon(t,x) = \sum_{j=1}^{m} Z_{\delta m j}^\varepsilon(t) \psi_j(x) \tag{2.22}$$

such that

$$(\frac{\partial \bar v^\varepsilon_{\delta m}}{\partial t}, \Theta_i) + a(\bar v^\varepsilon_{\delta m}, \Theta_i) + B(\bar v^\varepsilon_{\delta m}, \bar v^\varepsilon_{\delta m}, \Theta_i) + \frac{1}{2}(v^\varepsilon_{\delta m} \mathrm{div}\, v^\varepsilon_{\delta m}\,,\, \Phi_i)$$
$$+\frac{1}{2}(Z_{\delta m} \mathrm{div}\, v^\varepsilon_{\delta m}\,,\, \psi_i) + \frac{1}{\delta}(\mathrm{div} v^\varepsilon_{\delta m}\,,\, \mathrm{div}\Phi_i) = (\mathcal{F}(\bar v^\varepsilon_{\delta m}), \Theta_i) - \mathcal{R}(\bar v^\varepsilon_{\delta m}, \Theta_i)$$

(2.23) $\forall \Theta_i = (\Phi_i, \psi_i),\ 1 \le i \le m,$

(2.24) $\bar v^\varepsilon_{\delta m}(0) = \bar v^\varepsilon_{0m}.$

By taking $\psi_i = 0$ in (2.23) we deduce

$$(\frac{\partial v^\varepsilon_{\delta m}}{\partial t}, \Phi_i) + (\nu+\nu_r)(\nabla v^\varepsilon_{\delta m}, \nabla \Phi_i) + b(v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}, \Phi_i) + \frac{1}{2}(v^\varepsilon_{\delta m} \mathrm{div} v^\varepsilon_{\delta m}\,,\, \Phi_i)$$
$$+\frac{1}{\delta}(\mathrm{div}\, v^\varepsilon_{\delta m}\,,\, \mathrm{div}\Phi_i) = (\mathcal{F}_1(v^\varepsilon_{\delta m}), \Phi_i) + 2\nu_r(\mathrm{rot}\, Z^\varepsilon_{\delta m}, \Phi_i) \qquad 1 \le i \le m, \tag{2.25}$$
$$v^\varepsilon_{\delta m}(0) = v^\varepsilon_{0m}, \tag{2.26}$$

and by taking $\Phi_i = 0$ in (2.23) we deduce

$$(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \psi_i) + \alpha(\nabla Z^\varepsilon_{\delta m}, \nabla \psi_i) + b_1(v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}, \psi_i) + \frac{1}{2}(Z^\varepsilon_{\delta m} \mathrm{div}\, v^\varepsilon_{\delta m}\,, \psi_i) = (\mathcal{F}_2(v^\varepsilon_{\delta m}), \psi_i)$$
$$+2\nu_r(\mathrm{rot}\, v^\varepsilon_{\delta m}, \psi_i) - 4\nu_r(Z^\varepsilon_{\delta m}\,,\, \psi_i) \qquad 1 \le i \le m, \tag{2.27}$$
$$Z^\varepsilon_\delta(0) = Z^\varepsilon_{0m}, \tag{2.28}$$

where

$$(\mathcal{F}_1(v^\varepsilon_{\delta m}), \Phi_i) = -(\nu+\nu_r)(\nabla U^\varepsilon e_1, \nabla \Phi_i) - b(U^\varepsilon e_1\,,\, v^\varepsilon_{\delta m}\,,\, \Phi_i) - b(v^\varepsilon_{\delta m}\,,\, U^\varepsilon_\delta e_1\,,\, \Phi_i)$$
$$+2\nu_r(\frac{\partial W^\varepsilon}{\partial z_2} e_1\,,\, \Phi_i) - (\frac{\partial U^\varepsilon}{\partial t} e_1\,,\, \Phi_i) + (f^\varepsilon\,,\, \Phi_i), \tag{2.29}$$

and

$$(\mathcal{F}_2(v^\varepsilon_{\delta m}), \psi_i) = -\alpha(\nabla W^\varepsilon, \nabla \psi_i) - b_1(U^\varepsilon e_1\,,\, Z^\varepsilon\,,\, \psi_i) - b_1(v^\varepsilon_{\delta m}\,,\, W^\varepsilon\,,\, \psi_i)$$
$$-2\nu_r(\frac{\partial U^\varepsilon}{\partial z_2}\,,\, \psi_i) - 4\nu_r(W^\varepsilon\,,\, \psi_i) - (\frac{\partial W^\varepsilon}{\partial t}\,,\, \psi_i) + (g^\varepsilon\,,\, \psi_i). \tag{2.30}$$

Taking (2.22) into account, we deduce from (2.25)-(2.30) a system of (nonlinear) differential equations for the unknown scalar functions $(v^\varepsilon_{\delta mi}, Z^\varepsilon_{\delta mi})_{1\le i\le m}$, which possesses an unique maximal solution in $(H^1(0,T_m))^m$ with $T_m \in (0,T]$.

In order to prove that this solution is defined on the whole time interval $[0,T]$, we will establish some a priori estimates for $v^\varepsilon_{\delta m}$ and $Z^\varepsilon_{\delta m}$, independently of m. More precisely, we multiply the two sides of (2.25) by $v^\varepsilon_{\delta mi}(t)$ and the two sides of (2.27) by $Z^\varepsilon_{\delta mi}(t)$, then we sum for i from 1 to m, to get, with $\|\cdot\| = \|\cdot\|_{L^2(\Omega^\varepsilon)}$, the following equations

$$\frac{1}{2}\frac{\partial}{\partial t}(\|v^\varepsilon_{\delta m}\|^2) + (\nu+\nu_r)\|\nabla v^\varepsilon_{\delta m}\|^2 + b(v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}) + \frac{1}{2}(v^\varepsilon_{\delta m} \mathrm{div} v^\varepsilon_{\delta m}\,,\, v^\varepsilon_{\delta m}) + \frac{1}{\delta}\|\mathrm{div} v^\varepsilon_{\delta m}\|^2$$
$$= (\mathcal{F}_1(v^\varepsilon_{\delta m}), v^\varepsilon_{\delta m}) + 2\nu_r(\mathrm{rot}\, Z^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}), \tag{2.31}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(\||Z^\varepsilon_{\delta m}\|^2) + \alpha\|\nabla Z^\varepsilon_{\delta m}\|^2 + b_1(v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}) + \frac{1}{2}(Z^\varepsilon_{\delta m} \mathrm{div}\, v^\varepsilon_{\delta m}\,, Z^\varepsilon_{\delta m}) = (\mathcal{F}_2(v^\varepsilon_{\delta m}), Z^\varepsilon_{\delta m})$$
$$+2\nu_r(\mathrm{rot}\, v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}) - 4\nu_r(Z^\varepsilon_{\delta m}\,,\, Z^\varepsilon_{\delta m}). \tag{2.32}$$

By integration by parts and using the boundary conditions (2.12)-(2.14), we obtain that

$$b(v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}) + b_1(v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}) + \frac{1}{2}(v^\varepsilon_{\delta m}\mathrm{div} v^\varepsilon_{\delta m}\,,\, v^\varepsilon_{\delta m}) + \frac{1}{2}(Z^\varepsilon_{\delta m}\mathrm{div}\, v^\varepsilon_{\delta m}\,, Z^\varepsilon_{\delta m}) = 0,$$

and

$$b(U^\varepsilon e_1\,,\, v^\varepsilon_{\delta m}\,,\, v^\varepsilon_{\delta m}) + b_1(U^\varepsilon e_1\,,\, Z^\varepsilon_{\delta m}\,,\, Z^\varepsilon_{\delta m}) = 0.$$

Thus by the addition of (2.31) and (2.32) we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(\|v^\varepsilon_{\delta m}\|^2 + \||Z^\varepsilon_{\delta m}\|^2) + (\nu+\nu_r)\|\nabla v^\varepsilon_{\delta m}\|^2 + \alpha\|\nabla Z^\varepsilon_{\delta m}\|^2 + \frac{1}{\delta}\|\mathrm{div} v^\varepsilon_{\delta m}\|^2 = \Xi \tag{2.33}$$

with

$$\begin{aligned}\Xi &= 2\nu_r(\mathrm{rot}\, Z^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}) + 2\nu_r(\mathrm{rot}\, v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}) - 4\nu_r\|Z^\varepsilon_{\delta m}\|^2 - (\nu+\nu_r)(\nabla U^\varepsilon e_1, \nabla v^\varepsilon_{\delta m})\\
&\quad -\alpha(\nabla W^\varepsilon, \nabla Z^\varepsilon_{\delta m}) - b(v^\varepsilon_{\delta m}\,,\, U^\varepsilon e_1\,,\, v^\varepsilon_{\delta m}) - b_1(v^\varepsilon_{\delta m}\,,\, W^\varepsilon\,,\, Z^\varepsilon_{\delta m}) + 2\nu_r(\mathrm{rot}\, W^\varepsilon\,,\, v^\varepsilon_{\delta m})\\
&\quad +2\nu_r(\mathrm{rot}\, U^\varepsilon e_1\,,\, Z^\varepsilon_{\delta m}) - 4\nu_r(W^\varepsilon\,,\, Z^\varepsilon_{\delta m}) - (\frac{\partial U^\varepsilon e_1}{\partial t}\,,\, v^\varepsilon_{\delta m}) - (\frac{\partial W^\varepsilon}{\partial t}\,,\, Z^\varepsilon_{\delta m})\\
&\quad +(f^\varepsilon\,,\, v^\varepsilon_{\delta m}) + (g^\varepsilon\,,\, Z^\varepsilon_{\delta m}).\end{aligned}$$

Using Young's inequality we have

$$2\nu_r|(\mathrm{rot}\, Z^\varepsilon_{\delta m}, v^\varepsilon_{\delta m})| \le 2\nu_r\|\mathrm{rot}\, Z^\varepsilon_{\delta m}\|\|v^\varepsilon_{\delta m}\| \le \frac{\alpha}{4}\|\nabla Z^\varepsilon_{\delta m}\|^2 + \frac{4\nu_r^2}{\alpha}\|v^\varepsilon_{\delta m}\|^2,$$

$$\begin{aligned}2\nu_r|(\mathrm{rot}\, v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m})| &\le 2\nu_r\|\mathrm{rot}\, v^\varepsilon_{\delta m}\|\|Z^\varepsilon_{\delta m}\| \le \frac{\nu_r}{4}\|\mathrm{rot}\, v^\varepsilon_{\delta m}\|^2 + 4\nu_r\|Z^\varepsilon_{\delta m}\|^2\\
&\le \frac{\nu_r}{2}\|\nabla v^\varepsilon_{\delta m}\|^2 + 4\nu_r\|Z^\varepsilon_{\delta m}\|^2,\end{aligned}$$

$$(\nabla U^\varepsilon e_1, \nabla v^\varepsilon_{\delta m}) \le \frac{1}{2}\|\nabla(v^\varepsilon_{\delta m})\|^2 + \frac{1}{2}\|\frac{\partial U^\varepsilon}{\partial z_2}\|^2,$$

$$(\nabla W^\varepsilon, \nabla Z^\varepsilon_{\delta m}) \le \frac{1}{4}\|\nabla Z^\varepsilon_{\delta m}\|^2 + \|\frac{\partial W^\varepsilon}{\partial z_2}\|^2,$$

$$b(v^\varepsilon_{\delta m}\,,\, U^\varepsilon e_1\,,\, v^\varepsilon_{\delta m}) \le \|(v^\varepsilon_{\delta m})_2\|\|\frac{\partial U^\varepsilon}{\partial z_2}\|_\infty\|(v^\varepsilon_{\delta m})_1\| \le \|\frac{\partial U^\varepsilon}{\partial z_2}\|_\infty\|v^\varepsilon_{\delta m}\|^2,$$

$$b_1(v^\varepsilon_{\delta m}\,,\, W^\varepsilon\,,\, Z^\varepsilon_{\delta m}) \le \|(v^\varepsilon_{\delta m})_2\|\|\frac{\partial W^\varepsilon}{\partial z_2}\|_\infty\|Z^\varepsilon_{\delta m}\| \le \frac{1}{2}\|\frac{\partial W^\varepsilon}{\partial z_2}\|_\infty\left(\|v^\varepsilon_{\delta m}\|^2 + \|Z^\varepsilon_{\delta m}\|^2\right),$$

$$\begin{aligned}&2\nu_r(\mathrm{rot}\, W^\varepsilon\,,\, v^\varepsilon_{\delta m}) + 2\nu_r(\mathrm{rot}\, U^\varepsilon e_1\,,\, Z^\varepsilon_{\delta m}) - 4\nu_r(W^\varepsilon\,,\, Z^\varepsilon_{\delta m})\\
&= +2\nu_r\left(\frac{\partial W^\varepsilon}{\partial z_2}, (v^\varepsilon_{\delta m})_1\right) - 2\nu_r\left(\frac{\partial U^\varepsilon}{\partial z_2}, Z^\varepsilon_{\delta m}\right) - 4\nu_r(W^\varepsilon, Z^\varepsilon_{\delta m})\\
&\le \nu_r\|v^\varepsilon_{\delta m}\|^2 + 2\nu_r\|Z^\varepsilon_{\delta m}\|^2 + \nu_r\|\frac{\partial W^\varepsilon}{\partial z_2}\|^2 + \nu_r\|\frac{\partial U^\varepsilon}{\partial z_2}\|^2 + 4\nu_r\|W^\varepsilon\|^2.\end{aligned}$$

So we have

$$
\begin{aligned}
\Xi \leq (\frac{\nu}{2}+\nu_r)\|\nabla v^\varepsilon_{\delta m}\|^2 &+ \frac{\alpha}{2}\|\nabla Z^\varepsilon_{\delta m}\|^2 + \left(2+4\nu_r+\frac{1}{2}\|\frac{\partial W^\varepsilon(t)}{\partial z_2}\|_\infty\right)\|Z^\varepsilon_{\delta m}\|^2 \\
&+\left(2+\nu_r+\frac{4\nu_r^2}{\alpha}+\|\frac{\partial U^\varepsilon(t)}{\partial z_2}\|_\infty+\frac{1}{2}\|\frac{\partial W^\varepsilon(t)}{\partial z_2}\|_\infty\right)\|v^\varepsilon_{\delta m}\|^2 \\
&+\frac{(\nu+\nu_r)}{2}\|\frac{\partial U^\varepsilon(t)}{\partial z_2}\|^2+\alpha\|\frac{\partial W^\varepsilon(t)}{\partial z_2}\|^2+\nu_r\|\frac{\partial W^\varepsilon(t)}{\partial z_2}\|^2+\nu_r\|\frac{\partial U^\varepsilon(t)}{\partial z_2}\|^2 \\
&+4\nu_r\|W^\varepsilon(t)\|^2+\left\|\frac{\partial U^\varepsilon(t)}{\partial t}\right\|^2+\left\|\frac{\partial W^\varepsilon(t)}{\partial t}\right\|^2+\|f^\varepsilon(t)\|^2+\|g^\varepsilon(t)\|^2. \qquad (2.34)
\end{aligned}
$$

From (2.33)-(2.34), we get

$$
\frac{1}{2}\frac{\partial}{\partial t}([\bar v^\varepsilon_{\delta m}]^2)+\frac{k}{2}[[\bar v^\varepsilon_{\delta m}]]^2+\frac{1}{\delta}\|\mathrm{div}\, v^\varepsilon_{\delta m}\|^2 \leq A(t)[\bar v^\varepsilon_{\delta m}]^2+B(t), \qquad (2.35)
$$

where $k=\min\{\nu,\alpha\}$ and A and B belong to $L^1(0,T)$ such that $A(t)\geq 2$ and $B(t)\geq 0$ almost everywhere on $[0,T]$. Moreover A and B depend neither on m nor on δ.

For any $t\in(0,T_m)$ we can integrate the inequality (2.35) over $[0,t]$: we obtain

$$
\begin{aligned}
[\bar v^\varepsilon_{\delta m}(t)]^2+k\int_0^t[[\bar v^\varepsilon_{\delta m}(s)]]^2ds+\frac{2}{\delta}\int_0^t\|\mathrm{div}\, v^\varepsilon_{\delta m}(s)\|^2ds &\leq [\bar v^\varepsilon_0]^2 \\
&+2\int_0^t A(t)[v^{\bar\varepsilon}_{\delta m}(s)]^2ds+2\mathcal{B}, \qquad (2.36)
\end{aligned}
$$

with $\mathcal{B}=\int_0^T B(t)dt$. So by Grönwall's inequality, we deduce first that

$$
[\bar v^\varepsilon_{\delta m}(t)]^2 \leq ([\bar v^\varepsilon_0]^2+2\mathcal{B})e^{2\mathcal{A}} \quad \text{with} \quad \mathcal{A}=\int_0^T A(t)dt.
$$

Thus $\bar v^\varepsilon_{\delta m}$ is defined on the whole interval $[0,T]$ and

$$
\sup_{t\in[0,T]}[\bar v^\varepsilon_{\delta m}(t)]^2 \leq C. \qquad (2.37)
$$

Then from (2.36) and (2.37), we deduce

$$
\frac{1}{\delta}\int_0^T\|\mathrm{div}\, v^\varepsilon_{\delta m}(t)\|^2dt \leq C, \qquad \int_0^T[[\bar v^\varepsilon_{\delta m}(t)]]^2dt \leq C, \qquad (2.38)
$$

where here and in what follows $C's$ denotes various constants which depend neither on m nor on δ.

We need now to look at the time derivative of $v^\varepsilon_{\delta m}$ and $Z^\varepsilon_{\delta m}$. Let $\Theta^\varepsilon=(\varphi^\varepsilon,\psi^\varepsilon)\in (H^1_0(\Omega^\varepsilon))^2\times H^1_0(\Omega^\varepsilon)\subset V^\varepsilon\times H^{1,\varepsilon}$. There exists a sequence $(q^\varepsilon_i,k^\varepsilon_i)_{i\geq 1}$ in $\mathbb{R}^2$ such that

$$
\Theta^\varepsilon_p=(\varphi^\varepsilon_p,\psi^\varepsilon_p)\to(\varphi^\varepsilon,\psi^\varepsilon) \quad \text{strongly in } V^\varepsilon\times H^{1,\varepsilon}
$$

with

$$
\varphi^\varepsilon_p=\sum_{i=1}^p q^\varepsilon_i\Phi_i, \quad \psi^\varepsilon_p=\sum_{i=1}^p k^\varepsilon_i\psi_i \qquad \forall p\geq 1.
$$

Let $p \geq m$. Reminding that $(\Phi_i)_{i\geq 1}$ and $(\psi_i)_{i\geq 1}$ are orthogonal bases of $(L^2(\Omega^\varepsilon))^2$ and $L^2(\Omega^\varepsilon)$ respectively, we get

$$\left(\frac{\partial v^\varepsilon_{\delta m}}{\partial t}, \varphi^\varepsilon_p\right) = \sum_{j=1}^m (v^\varepsilon_{\delta m j})'(t)(\Phi_j, \varphi^\varepsilon_p) = \sum_{j=1}^m (v^\varepsilon_{\delta m j})'(t)(\Phi_j, \varphi^\varepsilon_m) = \left(\frac{\partial v^\varepsilon_{\delta m}}{\partial t}, \varphi^\varepsilon_m\right),$$

and

$$\left(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \psi^\varepsilon_p\right) = \sum_{j=1}^m (Z^\varepsilon_{\delta m j})'(t)(\Phi_j, \psi^\varepsilon_p) = \sum_{j=1}^m (Z^\varepsilon_{\delta m j})'(t)(\Phi_j, \psi^\varepsilon_m) = \left(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \psi^\varepsilon_m\right).$$

Since $\frac{\partial \bar{v}^\varepsilon_{\delta m}}{\partial t} \in L^2(0,T; V^\varepsilon \times H^{1,\varepsilon})$, we can pass to the limit as p tends to $+\infty$ i.e

$$\left(\frac{\partial v^\varepsilon_{\delta m}}{\partial t}, \varphi^\varepsilon\right) = \left(\frac{\partial v^\varepsilon_{\delta m}}{\partial t}, \varphi^\varepsilon_m\right), \quad \left(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \psi^\varepsilon\right) = \left(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \varphi^\varepsilon_m\right).$$

Then, by using Green's formula and (2.25)

$$\begin{aligned}\left(\frac{\partial v^\varepsilon_{\delta m}}{\partial t}, \varphi^\varepsilon\right) = \Big((\nu+\nu_r)\Delta v^\varepsilon_{\delta m} - (v^\varepsilon_{\delta m}\cdot\nabla)v^\varepsilon_{\delta m} - \frac{1}{2}v^\varepsilon_{\delta m}\operatorname{div} v^\varepsilon_{\delta m} \\ + \mathcal{F}_1(v^\varepsilon_{\delta m}) + 2\nu_r \operatorname{rot} Z^\varepsilon_{\delta m} + \frac{1}{\delta}\nabla(\operatorname{div} v^\varepsilon_{\delta m}), \varphi^\varepsilon_m\Big),\end{aligned} \tag{2.39}$$

and from (2.27)

$$\begin{aligned}\left(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \psi^\varepsilon\right) = \Big(\alpha\Delta Z^\varepsilon_{\delta m} - (v^\varepsilon_{\delta m}\cdot\nabla)Z^\varepsilon_{\delta m} - \frac{1}{2}Z^\varepsilon_{\delta m}\operatorname{div} v^\varepsilon_{\delta m} + \mathcal{F}_2(v^\varepsilon_{\delta m}) + 2\nu_r\operatorname{rot} v^\varepsilon_{\delta m} \\ -4\nu_r Z^\varepsilon_{\delta m}, \psi^\varepsilon_m\Big)\end{aligned} \tag{2.40}$$

and from (2.29)

$$\mathcal{F}_1(v^\varepsilon_{\delta m}) = (\nu+\nu_r)\frac{\partial^2 U^\varepsilon}{\partial z_2^2}e_1 - U^\varepsilon\frac{\partial v^\varepsilon_{\delta m}}{\partial z_1} - (v^\varepsilon_{\delta m})_2\frac{\partial U^\varepsilon}{\partial z_2}e_1 + 2\nu_r\frac{\partial W^\varepsilon}{\partial z_2}e_1 - \frac{\partial U^\varepsilon}{\partial t}e_1 + f^\varepsilon,$$

and from (2.30)

$$\mathcal{F}_2(v^\varepsilon_{\delta m}) = \alpha\frac{\partial^2 W^\varepsilon}{\partial z_2^2} - U^\varepsilon\frac{\partial Z^\varepsilon_{\delta m}}{\partial z_1} - (v^\varepsilon_{\delta m})_2\frac{\partial W^\varepsilon}{\partial z_2} + 2\nu_r\frac{\partial U^\varepsilon}{\partial z_2} - 4\nu_r W^\varepsilon - \frac{\partial W^\varepsilon}{\partial t} + g^\varepsilon.$$

As $v^\varepsilon_{\delta m}$ is bounded in $L^2(0,T;(H^1(\Omega^\varepsilon)^2))$ independently of m and δ, then $\Delta v^\varepsilon_{\delta m}$ and $\nabla(\operatorname{div} v^\varepsilon_{\delta m})$ are also bounded in $L^2(0,T;(H^{-1}(\Omega^\varepsilon))^2)$ independently of m and δ. Similarly, since $Z^\varepsilon_{\delta m}$ is bounded in $L^2(0,T;H^1(\Omega^\varepsilon))$ independently of m and δ, then $\operatorname{rot} Z^\varepsilon_{\delta m}$ is also bounded in $L^2(0,T;(L^2(\Omega^\varepsilon))^2)$ independently of m and δ. By assumption, $f^\varepsilon \in (L^2((0,T)\times\Omega^\varepsilon)^2$, $g^\varepsilon \in L^2((0,T)\times\Omega^\varepsilon)$, and from Lemma 2.1, U^ε and W^ε belong to $H^1(0,T)\times\mathcal{D}((-\infty,h_m))$. Thus we infer that $\mathcal{F}_1(v^\varepsilon_{\delta m})$ and $\mathcal{F}_2(v^\varepsilon_{\delta m})$ are bounded in $L^2(0,T;(L^2(\Omega^\varepsilon))^2)$ and $L^2(0,T;L^2(\Omega^\varepsilon)$, independently of m and δ. Moreover let $\varphi \in (H^1(\Omega^\varepsilon))^2$, we have

$$|((v^\varepsilon_{\delta m}\cdot\nabla)v^\varepsilon_{\delta m}\,,\,\varphi)| \leq \|v^\varepsilon_{\delta m}\|_{L^3(\Omega^\varepsilon)}\|\nabla v^\varepsilon_{\delta m}\|_{L^2(\Omega^\varepsilon)}\|\varphi\|_{L^6(\Omega^\varepsilon)}.$$

Using now the classical inequality

$$\|u\|_{L^3(\Omega^\varepsilon)} \leq \|u\|^{1/2}_{L^2(\Omega^\varepsilon)}\|u\|^{1/2}_{L^6(\Omega^\varepsilon)} \quad \forall u \in L^6(\Omega^\varepsilon),$$

and the continuous injection of $H^1(\Omega^\varepsilon)$ in $L^6(\Omega^\varepsilon)$, there exists a constant C such that

$$|((v^\varepsilon_{\delta m}\cdot\nabla)v^\varepsilon_{\delta m}\,,\,\varphi)|\leq\Big(C\|v^\varepsilon_{\delta m}\|^{1/2}_{L^2(\Omega^\varepsilon)}\|\nabla v^\varepsilon_{\delta m}\|^{3/2}_{L^2(\Omega^\varepsilon)}\Big)\|\varphi\|_{H^1(\Omega^\varepsilon)}.$$

So we get

$$\|(v^\varepsilon_{\delta m}\cdot\nabla)v^\varepsilon_{\delta m}\|_{(H^1(\Omega^\varepsilon))'}\leq C\|v^\varepsilon_{\delta m}\|^{1/2}_{L^2(\Omega^\varepsilon)}\|\nabla v^\varepsilon_{\delta m}\|^{3/2}_{L^2(\Omega^\varepsilon)}$$

then

$$\begin{aligned}\int_0^T\|(v^\varepsilon_{\delta m}\cdot\nabla)v^\varepsilon_{\delta m}\|^{4/3}_{(H^1(\Omega^\varepsilon))'}dt&\leq C^{4/3}\int_0^T\|v^\varepsilon_{\delta m}\|^{2/3}_{L^2(\Omega^\varepsilon)}\|\nabla v^\varepsilon_{\delta m}\|^2_{L^2(\Omega^\varepsilon)}\\&\leq C^{4/3}\|v^\varepsilon_{\delta m}\|^{2/3}_{L^\infty(0,T;L^2(\Omega^\varepsilon))}\|\nabla v^\varepsilon_{\delta m}\|^2_{L^2((0,T)\times\Omega^\varepsilon)}.\end{aligned}$$

With the same arguments, we deduce similar result for $v^\varepsilon_{\delta m}\mathrm{div}\,v^\varepsilon_{\delta m}$, $(v^\varepsilon_{\delta m}\cdot\nabla)Z^\varepsilon_{\delta m}$ and $Z^\varepsilon_{\delta m}\mathrm{div}\,v^\varepsilon_{\delta m}$. Finally, recalling that $(\Phi_i)_{i\geq1}$ and $(\psi_i)_{i\geq1}$ are H^1-orthonormal, we get

$$\|\varphi^\varepsilon_m\|_{(H^1(\Omega^\varepsilon))^2}\leq\|\varphi^\varepsilon\|_{(H^1(\Omega^\varepsilon))^2},\quad\|\psi^\varepsilon_m\|_{H^1(\Omega^\varepsilon)}\leq\|\psi^\varepsilon\|_{H^1(\Omega^\varepsilon)}\quad\forall m\geq1.$$

So from (2.39) and (2.40) we see that there exists a constant C such that

$$\|\frac{\partial v^\varepsilon_{\delta m}}{\partial t}\|_{L^{4/3}(0,T;(H^{-1}(\Omega^\varepsilon))^2)}\leq C,\quad\|\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}\|_{L^{4/3}(0,T;H^{-1}(\Omega^\varepsilon))}\leq C.\tag{2.41}$$

From the estimates (2.37)-(2.38) we infer that there exists a subsequence (denoted also by) $\bar v^\varepsilon_{\delta m}$ such that

$$\bar v^\varepsilon_{\delta m}\rightharpoonup\bar v^\varepsilon_\delta\quad\text{in }L^2(0,T;V^\varepsilon)\times L^2(0,T;H^{1,\varepsilon})\quad\text{weakly for }m\to+\infty,\tag{2.42}$$

$$\bar v^\varepsilon_{\delta m}\rightharpoonup\bar v^\varepsilon_\delta\quad\text{in }L^\infty(0,T;H^\varepsilon)\times L^\infty(0,T;H^{0,\varepsilon})\quad\text{weak star for }m\to+\infty,\tag{2.43}$$

and from (2.41), by Aubin's compactness theorem A.11 in [15], there are two subsequences (denoted also by) $v^\varepsilon_{\delta m}$, $Z^\varepsilon_{\delta m}$ satisfying for $m\to+\infty$ the following strong convergence

$$v^\varepsilon_{\delta m}\to v^\varepsilon_\delta\text{ in }L^2(0,T;(L^4(\Omega^\varepsilon))^2),\quad Z^\varepsilon_{\delta m}\to Z^\varepsilon_\delta\text{ in }L^2(0,T;L^4(\Omega^\varepsilon)).\tag{2.44}$$

In order to pass to the limit as $m\to+\infty$, we remind that for any $\Theta^\varepsilon=(\varphi^\varepsilon,\psi^\varepsilon)\in V^\varepsilon\times H^{1,\varepsilon}$, there exists a sequence $(q^\varepsilon_i,k^\varepsilon_i)_{i\geq1}$ in $\mathbb{R}^2$ such that

$$\Theta^\varepsilon_m=(\varphi^\varepsilon_m,\psi^\varepsilon_m)\to(\varphi^\varepsilon,\psi^\varepsilon)\quad\text{strongly in }V^\varepsilon\times H^{1,\varepsilon}$$

with

$$\varphi^\varepsilon_m=\sum_{i=1}^m q^\varepsilon_i\Phi_i,\quad\psi^\varepsilon_m=\sum_{i=1}^m k^\varepsilon_i\psi_i\qquad\forall m\geq1.$$

We multiply first the two sides of (2.25) by q^ε_i then we sum for $i=1$ to m, and we multiply the two sides of (2.27) by k^ε_i then we sum also for $i=1$ to m, we obtain

$$\begin{aligned}&(\frac{\partial v^\varepsilon_{\delta m}}{\partial t},\varphi^\varepsilon_m)+(\nu+\nu_r)(\nabla v^\varepsilon_{\delta m},\nabla\varphi^\varepsilon_m)+b(v^\varepsilon_{\delta m},v^\varepsilon_{\delta m},\varphi^\varepsilon_m)+\frac12(v^\varepsilon_{\delta m}\mathrm{div}v^\varepsilon_{\delta m}\,,\,\varphi^\varepsilon_m)\\&\quad+\frac1\delta(\mathrm{div}v^\varepsilon_{\delta m}\,,\,\mathrm{div}\varphi^\varepsilon_m)=(\mathcal F_1(v^\varepsilon_{\delta m}),\varphi^\varepsilon_m)+2\nu_r(\mathrm{rot}\ Z^\varepsilon_{\delta m},\varphi^\varepsilon_m),\end{aligned}\tag{2.45}$$

$$v^\varepsilon_{\delta m}(0)=v^\varepsilon_{0m},\tag{2.46}$$

and

$$(\frac{\partial Z^\varepsilon_{\delta m}}{\partial t}, \psi^\varepsilon_m) + \alpha(\nabla Z^\varepsilon_{\delta m}, \nabla \psi^\varepsilon_m) + b_1(v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}, \psi^\varepsilon_m) + \frac{1}{2}(Z^\varepsilon_{\delta m} \operatorname{div} v^\varepsilon_{\delta m}, \psi^\varepsilon_m)$$

$$= (\mathcal{F}_2(v^\varepsilon_{\delta m}), \psi^\varepsilon_m) + 2\nu_r(\operatorname{rot} v^\varepsilon_{\delta m}, \psi^\varepsilon_m) - 4\nu_r(Z^\varepsilon_{\delta m}, \psi^\varepsilon_m), \tag{2.47}$$

$$Z^\varepsilon_\delta(0) = Z^\varepsilon_{0m}. \tag{2.48}$$

Let $\theta \in \mathcal{D}(0,T)$, we multiply (2.45) and (2.47) by $\theta(t)$ and we integrate over $[0,T]$. We get

$$-\int_0^T (\bar{v}^\varepsilon_{\delta m}(t), \Theta^\varepsilon_m)\theta'(t)dt + \int_0^T \{a(\bar{v}^\varepsilon_{\delta m}, \Theta^\varepsilon_m) + B(\bar{v}^\varepsilon_{\delta m}, \bar{v}^\varepsilon_{\delta m}, \Theta^\varepsilon_m)\}\,\theta(t)dt$$

$$+\frac{1}{\delta}\int_0^T (\operatorname{div} v^\varepsilon_{\delta m}, \operatorname{div}\varphi^\varepsilon_m)\theta(t)dt + \frac{1}{2}\int_0^T \{v^\varepsilon_{\delta m}\operatorname{div} v^\varepsilon_{\delta m}, \varphi^\varepsilon_m) + (Z^\varepsilon_\delta \operatorname{div} v^\varepsilon_{\delta m}, \psi^\varepsilon_m)\}\,\theta(t)dt$$

$$= \int_0^T \{(\mathcal{F}(\bar{v}^\varepsilon_{\delta m}), \Theta^\varepsilon_m) - \mathcal{R}(\bar{v}^\varepsilon_{\delta m}, \Theta^\varepsilon_m)\}\,\theta(t)dt. \tag{2.49}$$

Using the convergences (2.42)-(2.43), we can now pass easily to the limit in all terms of (2.49) except for the nonlinear terms

$$\int_0^T B(\bar{v}^\varepsilon_{\delta m}, \bar{v}^\varepsilon_{\delta m}, \Theta^\varepsilon_m)\theta(t)dt = \int_0^T b(v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}, \varphi^\varepsilon_m)\theta(t)dt + \int_0^T b_1(v^\varepsilon_{\delta m}, Z^\varepsilon_\delta, \psi^\varepsilon_m)\theta(t)dt$$

and

$$\int_0^T \{v^\varepsilon_{\delta m}\operatorname{div} v^\varepsilon_{\delta m}, \varphi^\varepsilon_m) + (Z^\varepsilon_\delta \operatorname{div} v^\varepsilon_{\delta m}, \psi^\varepsilon_m)\}\,\theta(t)dt.$$

We have first

$$\int_0^T b(v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}, \varphi^\varepsilon_m)\theta(t)dt = -\int_0^T b(v^\varepsilon_{\delta m}, \varphi^\varepsilon_m, v^\varepsilon_{\delta m})\theta(t)dt - \int_0^T (\operatorname{div} v^\varepsilon_{\delta m}, \varphi^\varepsilon_m \cdot v^\varepsilon_{\delta m})\theta(t)dt$$

$$+\int_0^T \int_{\partial\Omega^\varepsilon} (\varphi^\varepsilon_m \cdot v^\varepsilon_{\delta m})(v^\varepsilon_{\delta m}\cdot n)\theta(t)d\sigma dt. \tag{2.50}$$

Using the boundary conditions (2.12)-(2.14), we obtain that the last integral is equal to zero, then for the first and the second integrals we use the strong convergence (2.44). So we get

$$\int_0^T b(v^\varepsilon_{\delta m}, v^\varepsilon_{\delta m}, \varphi^\varepsilon_m)\theta(t)dt \to \int_0^T b(v^\varepsilon_\delta, v^\varepsilon_\delta, \varphi^\varepsilon)\theta(t)dt \text{ for } m \to +\infty.$$

Similarly

$$\int_0^T b(v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}, \psi^\varepsilon_m)\theta(t)dt = -\int_0^T b(v^\varepsilon_{\delta m}, \psi^\varepsilon_m, Z^\varepsilon_{\delta m})\theta(t)dt - \int_0^T (\operatorname{div} v^\varepsilon_{\delta m}, \psi^\varepsilon \cdot Z^\varepsilon_{\delta m})\theta(t)dt$$

$$+\int_0^T \int_{\partial\Omega^\varepsilon} (\psi^\varepsilon_m \cdot Z^\varepsilon_{\delta m})(v^\varepsilon_{\delta m}\cdot n)\theta(t)d\sigma dt. \tag{2.51}$$

Using the boundary conditions (2.12)-(2.14), we obtain that the last integral is equal to zero, then for the first and the second integrals we use the strong convergence (2.44). So we get

$$\int_0^T b(v^\varepsilon_{\delta m}, Z^\varepsilon_{\delta m}, \psi^\varepsilon_m)\theta(t)dt \to \int_0^T b(v^\varepsilon_\delta, \psi^\varepsilon, Z^\varepsilon_\delta)\theta(t)dt \text{ for } m \to +\infty.$$

We can now pass to the limit ($m \to +\infty$) in all terms of (2.49) to get

$$\int_0^T (v_\delta^\varepsilon, \Theta^\varepsilon)\theta'(t)dt = \int_0^T \left\{ a(\overline{v}_\delta^\varepsilon, \Theta^\varepsilon) + B(\overline{v}_\delta^\varepsilon, \overline{v}_\delta^\varepsilon, \Theta^\varepsilon) + \frac{1}{\delta}(\mathrm{div}\, v_\delta^\varepsilon\,,\, \mathrm{div}\varphi^\varepsilon) \right\} \theta(t)dt$$
$$+ \int_0^T \left\{ \frac{1}{2}(v_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\, \varphi^\varepsilon) + \frac{1}{2}(Z_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\, \psi^\varepsilon) - (\mathcal{F}(\overline{v}_\delta^\varepsilon), \Theta^\varepsilon) - \mathcal{R}(\overline{v}_\delta^\varepsilon, \Theta^\varepsilon) \right\} \theta(t)dt$$
$$\forall \Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon}, \tag{2.52}$$

that is $\overline{v}_\delta^\varepsilon$ satisfy (2.20) in $\mathcal{D}'(0,T)$ (distribution sense). Moreover as the two items between the brackets $\{\}$, in the right hand side of (2.52), are in $L^{4/3}(0,T)$, we deduce that (2.20) holds for almost every $t \in (0,T)$.

In the following we set

$$p_\delta^\varepsilon = -\frac{1}{\delta}\mathrm{div}\, v_\delta^\varepsilon,$$

then, rewrite (2.20) as follows

$$[\frac{\partial \overline{v}_\delta^\varepsilon}{\partial t}, \Theta^\varepsilon] + a(\overline{v}_\delta^\varepsilon, \Theta^\varepsilon) + B(\overline{v}_\delta^\varepsilon, \overline{v}_\delta^\varepsilon, \Theta^\varepsilon) + \frac{1}{2}\left\{(v_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\, \varphi^\varepsilon) + (Z_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\, \psi^\varepsilon)\right\}$$
$$-(p_\delta^\varepsilon\,,\, \mathrm{div}\varphi^\varepsilon) = (\mathcal{F}(\overline{v}_\delta^\varepsilon), \Theta^\varepsilon) - \mathcal{R}(\overline{v}_\delta^\varepsilon, \Theta^\varepsilon) \qquad \forall \Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon}. \tag{2.53}$$

The aim now is to pass to the limit for $\delta \to 0$ in (2.53). Reminding that the different constants C in (2.37)-(2.38) and (2.41) are independent of δ, the same estimates hold for $\overline{v}_\delta^\varepsilon$ i.e.

$$\sup_{t\in[0,T]} [\overline{v}_\delta^\varepsilon(t)]^2 \le C, \tag{2.54}$$

$$\int_0^T \|\mathrm{div}\, v_\delta^\varepsilon\|^2 dt \le C\delta, \quad \int_0^T [[\overline{v}_\delta^\varepsilon(t)]]^2 dt \le C, \tag{2.55}$$

and

$$\|\frac{\partial v_\delta^\varepsilon}{\partial t}\|_{L^{4/3}(0,T;H^{-1}(\Omega^\varepsilon)^2)} \le C, \quad \|\frac{\partial Z_\delta^\varepsilon}{\partial t}\|_{L^{4/3}(0,T;H^{-1}(\Omega^\varepsilon))} \le C. \tag{2.56}$$

Hence, there exists $\overline{v}^\varepsilon$ such that, possibly extracting a subsequence still denoted by $\overline{v}_\delta^\varepsilon$:

$$\overline{v}_\delta^\varepsilon \rightharpoonup \overline{v}^\varepsilon \quad \text{in } L^2(0,T;V^\varepsilon) \times L^2(0,T;H^{1,\varepsilon}) \quad \text{weakly for } \delta \to 0, \tag{2.57}$$

$$\overline{v}_\delta^\varepsilon \rightharpoonup \overline{v}^\varepsilon \quad \text{in } L^\infty(0,T;H^\varepsilon) \times L^\infty(0,T;H^{0,\varepsilon}) \quad \text{weak star for } \delta \to 0, \tag{2.58}$$

$$\mathrm{div}\, v_\delta^\varepsilon \to 0 \quad \text{in } L^2(0,T;L^2(\Omega^\varepsilon)) \quad \text{strongly for } \delta \to 0, \tag{2.59}$$

and

$$\overline{v}_\delta^\varepsilon \to \overline{v}^\varepsilon \text{ strongly in } L^2(0,T;(L^4(\Omega^\varepsilon)^2)) \times L^2(0,T;L^4(\Omega^\varepsilon)). \tag{2.60}$$

So from (2.57) and (2.59) we deduce

$$\operatorname{div} v^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \quad \text{a.e. in } (0,T). \tag{2.61}$$

We check now that p^ε_δ remains in a bounded subset of $H^{-1}(0,T;L^2_0(\Omega^\varepsilon))$. Reminding that $p^\varepsilon_\delta = -\frac{1}{\delta}\operatorname{div} v^\varepsilon_\delta$, we have $p^\varepsilon_\delta \in L^2(0,T;L^2_0(\Omega^\varepsilon))$. Now let us consider $\omega \in H^1_0(0,T;L^2_0(\Omega^\varepsilon))$, then (see [21] page 13-15) there exists $\varphi \in H^1_0(0,T;H^1_0(\Omega^\varepsilon)^2)$ such that

$$\begin{aligned}&\operatorname{div}\varphi(t) = \omega(t), \text{ and } \varphi(t) = P\omega(t),\\ &P \text{ is a linear continuous operator from } L^2_0(\Omega^\varepsilon) \text{ to } H^1_0(\Omega^\varepsilon)^2.\end{aligned}$$

The choice of $\Theta = (\varphi(t), 0)$ in (2.53), gives

$$\begin{aligned}\int_0^T (p^\varepsilon_\delta\,,\omega)dt &= \int_0^T \Big(-(v^\varepsilon_\delta, \frac{\partial\varphi}{\partial t}) + (\nu+\nu_r)(\nabla v^\varepsilon_\delta, \nabla\varphi)\Big)dt + \int_0^T b(v^\varepsilon_\delta, v^\varepsilon_\delta, \varphi)dt\\ &+\frac{1}{2}\int_0^T (v^\varepsilon_\delta \operatorname{div} v^\varepsilon_\delta\,,\varphi)dt - 2\nu_r\int_0^T(\operatorname{rot} Z^\varepsilon_\delta, \varphi)dt - \int_0^T (\mathcal{F}_1(v^\varepsilon_\delta),\varphi)dt,\end{aligned} \tag{2.62}$$

with

$$\begin{aligned}(\mathcal{F}_1(v^\varepsilon_\delta),\varphi) &= -(\nu+\nu_r)(\frac{\partial U^\varepsilon}{\partial z_2}, \frac{\partial \varphi_1}{\partial z_2}) - b(U^\varepsilon e_1, v^\varepsilon_\delta, \varphi) - b(v^\varepsilon_\delta, U^\varepsilon e_1, \varphi)\\ &+2\nu_r(\frac{\partial W^\varepsilon}{\partial z_2}, \varphi_1) - (\frac{\partial U^\varepsilon}{\partial t}, \varphi_1) + (f^\varepsilon, \varphi).\end{aligned} \tag{2.63}$$

Since $\omega \in H^1_0(0,T;L^2_0(\Omega^\varepsilon)) \subset L^\infty(0,T;L^2_0(\Omega^\varepsilon))$, with continuous injection, it follows that φ in $L^\infty(0,T;H^1_0(\Omega^\varepsilon)^2)$, and $\frac{\partial\varphi}{\partial t} \in L^2(0,T;H^1_0(\Omega^\varepsilon)^2)$, then also by the continuous injection of $H^1(\Omega^\varepsilon)$ in $L^4(\Omega^\varepsilon)$ we have

$$\begin{aligned}|\int_0^T b(v^\varepsilon_\delta, v^\varepsilon_\delta, \varphi)dt| &\le \|v^\varepsilon_\delta\|_{L^2(0,T;(L^4(\Omega^\varepsilon))^2)}\|v^\varepsilon_\delta\|_{L^2(0,T;H^1(\Omega^\varepsilon)^2)}\|\varphi\|_{L^\infty(0,T;(L^4(\Omega^\varepsilon))^2)}\\ &\le C^2\|v^\varepsilon_\delta\|^2_{L^2(0,T;H^1(\Omega^\varepsilon)^2)}\|\varphi\|_{H^1(0,T;H^1(\Omega^\varepsilon)^2)}.\end{aligned}$$

Similarly for the first term in the second line of (2.62). Therefore using (2.54)-(2.55) we get

$$|\int_0^T (p^\varepsilon_\delta\,,\omega)dt| \le C\|\varphi\|_{H^1(0,T;H^1(\Omega^\varepsilon)^2)} \quad \forall\varphi \in H^1_0(0,T;H^1_0(\Omega^\varepsilon)^2).$$

As $P:\omega(t) \mapsto \varphi(t)$ is a linear continuous operator from $L^2_0(\Omega^\varepsilon)$ to $H^1_0(\Omega^\varepsilon)^2$, there exists another constant C, independent of δ, such that

$$|\int_0^T (p^\varepsilon_\delta\,,\omega)dt| \le C\|\omega\|_{H^1_0(0,T;L^2_0(\Omega^\varepsilon))} \quad \forall\omega \in H^1(0,T;L^2(\Omega^\varepsilon)). \tag{2.64}$$

Let us take now $\omega \in H^1_0(0,T;L^2(\Omega^\varepsilon))$ arbitrary, we can apply (2.64) to

$$\tilde\omega = \omega - \frac{1}{meas(\Omega^\varepsilon)}\int_{\Omega^\varepsilon}\omega dz$$

which is in $H^1_0(0,T;L^2_0(\Omega^\varepsilon))$. But $p^\varepsilon_\delta \in L^2(0,T;L^2_0(\Omega^\varepsilon))$, so

$$\int_0^T (p^\varepsilon_\delta\,,\tilde\omega)dt = \int_0^T (p^\varepsilon_\delta\,,\omega)dt$$

and (2.64) remains valid for all $\omega \in H_0^1(0,T;L^2(\Omega^\varepsilon)$. Thus p_δ^ε remains in a bounded subset of $H^{-1}(0,T;L_0^2(\Omega^\varepsilon))$. It follows that there exists $p^\varepsilon \in H^{-1}(0,T;L_0^2(\Omega^\varepsilon))$ such that

$$p_\delta^\varepsilon \rightharpoonup p^\varepsilon \quad \text{in } H^{-1}(0,T;L^2(\Omega^\varepsilon)) \text{ weak.} \tag{2.65}$$

In order to pass to the limit as $\delta \to 0$, let $\theta \in \mathcal{D}(0,T)$, multiply (2.53) by $\theta(t)$ and integrate over $[0,T]$. We get

$$\begin{aligned}
&-\int_0^T (\bar{v}_\delta^\varepsilon(t),\Theta^\varepsilon)\theta'(t)dt + \int_0^T (a(\bar{v}_\delta^\varepsilon,\Theta^\varepsilon) + B(\bar{v}_\delta^\varepsilon,\bar{v}_\delta^\varepsilon,\Theta^\varepsilon))\,\theta(t)dt - \int_0^T (p_\delta^\varepsilon\,,\,\mathrm{div}\varphi^\varepsilon)\theta(t)dt\\
&+\frac{1}{2}\int_0^T \left\{(v_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\,\varphi^\varepsilon) + (Z_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\,\psi^\varepsilon)\right\}\theta(t)dt = \int_0^T \left\{(\mathcal{F}(\bar{v}_\delta^\varepsilon),\Theta^\varepsilon) - \mathcal{R}(\bar{v}_\delta^\varepsilon,\Theta^\varepsilon)\right\}\theta(t)dt\\
&\qquad \forall \Theta^\varepsilon = (\varphi^\varepsilon,\psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon}.
\end{aligned} \tag{2.66}$$

Using (2.57), (2.59), (2.60) and (2.65), then taking into account (2.50)-(2.51) for v_δ^ε and Z_δ^ε instead of $v_{\delta m}^\varepsilon$ and $Z_{\delta m}^\varepsilon$ for the nonlinear terms, we can now pass to the limit in all the terms of (2.66) to get

$$\begin{aligned}
&\int_0^T (v^\varepsilon,\Theta^\varepsilon)\theta'(t)dt = \int_0^T \left\{a(\bar{v}^\varepsilon,\Theta^\varepsilon) + B(\bar{v}^\varepsilon,\bar{v}^\varepsilon,\Theta^\varepsilon) - (p^\varepsilon\,,\,\mathrm{div}\varphi^\varepsilon)\right\}\theta(t)dt\\
&-\int_0^T \left\{(\mathcal{F}(\bar{v}^\varepsilon),\Theta^\varepsilon) - \mathcal{R}(\bar{v}^\varepsilon,\Theta^\varepsilon)\right\}\theta(t)dt \qquad \forall \Theta^\varepsilon = (\varphi^\varepsilon,\psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon},
\end{aligned}$$

that is $(\bar{v}^\varepsilon,p^\varepsilon)$ satisfy (2.17) in $\mathcal{D}'(0,T)$ (distribution sense). Moreover we can see also that (2.17) is satisfied for almost every $t \in (0,T)$.

Finally, by considering test-functions $\Theta^\varepsilon \in V_{div} \times H^{1,\varepsilon}$, we can prove the uniqueness of $(v^\varepsilon,Z^\varepsilon)$ and its continuity in time as in Theorem 2.2 [23]. Thus the proof of the existence and uniqueness of a solution of Problem (P^ε) is complete. ☐

3. A priori uniform estimates of $\bar{v}^\varepsilon$ and p^ε. The aim in this section is to establish uniform estimates with respect to ε for $\bar{v}^\varepsilon$ and p^ε, which will allow us to derive in the next sections the limit problem as ε tends to zero by using the two-scale convergence technique. More precisely we consider first the following scaling

$$x_1 = z_1, \quad \text{and} \quad x_2 = \frac{z_2}{\varepsilon}, \tag{3.1}$$

which transforms the domain Ω^ε into the domain

$$\Omega_\varepsilon = \left\{x = (x_1,x_2) \in \mathbb{R}^2 : \quad 0 < x_1 < L \qquad 0 < x_2 < h^\varepsilon(x_1) = h(x_1,\frac{x_1}{\varepsilon})\right\},$$

then we introduce a second scaling

$$y_1 = x_1, \quad \text{and} \quad y_2 = \frac{x_2}{h^\varepsilon(x_1)} = \frac{z_2}{\varepsilon h^\varepsilon(x_1)} \tag{3.2}$$

which transforms the domain Ω_ε into $\Omega = \{y = (y_1,y_2) \in \Gamma_0 \times (0,1)\}$. With the chain rule, we get easily the following relations

$$\begin{aligned}
\frac{\partial}{\partial z_2} = \frac{1}{\varepsilon h^\varepsilon(y_1)}\frac{\partial}{\partial y_2}, \quad \frac{\partial}{\partial z_1} &= \frac{\partial}{\partial y_1}\frac{\partial y_1}{\partial z_1} + \frac{\partial}{\partial y_2}\frac{\partial y_2}{\partial z_1} = \frac{\partial}{\partial y_1} + \left(-\frac{y_2}{h^\varepsilon(y_1)}\frac{\partial h^\varepsilon}{\partial y_1}\right)\frac{\partial}{\partial y_2}\\
&= \left(1, -\frac{y_2}{h^\varepsilon(y_1)}\frac{\partial h^\varepsilon}{\partial y_1}\right)\begin{pmatrix}\frac{\partial}{\partial y_1}\\[2ex] \frac{\partial}{\partial y_2}\end{pmatrix} = b_\varepsilon \cdot \nabla.
\end{aligned} \tag{3.3}$$

Now we define the functional setting in Ω: let $\Gamma_1 = \{(y_1, y_2) \in \overline{\Omega} : \quad y_2 = 1\}$ and

$$\tilde{V} = \{v \in \mathcal{C}^\infty(\overline{\Omega})^2 : \ v \text{ is L-periodic in } \ y_1, \ v_{|\Gamma_0} = 0, \ -\varepsilon(h^\varepsilon)'(y_1)v_1 + v_2 = 0 \text{ on } \Gamma_1\}$$

$$V = \text{closure of } \tilde{V} \text{ in } H^1(\Omega) \times H^1(\Omega)$$

$$\tilde{H^1} = \{Z \in \mathcal{C}^\infty(\overline{\Omega}) : \ Z \text{ is L-periodic in } \ y_1, \ Z = 0 \text{ on } \Gamma_0 \cup \Gamma_1\}$$

$$H = \text{closure of } \tilde{V} \text{ in } L^2(\Omega) \times L^2(\Omega), \qquad H^1 = \text{closure of } \tilde{H^1} \text{ in } H^1(\Omega),$$

$$H^0 = \text{closure of } \tilde{H^1} \text{ in } L^2(\Omega).$$

In order to avoid new notations, we have still denoted by v^ε, Z^ε and p^ε the unknown velocity, micro-rotation and pressure fields as functions of the rescaled variables (y_1, y_2) instead of (z_1, z_2). Similarly, we still denote the data by $\bar{f}^\varepsilon$ and $\bar{\xi}^\varepsilon$ considered now as functions of (y_1, y_2).

Let $\Theta = (\varphi, \psi) \in V \times H^1$ and let $\Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times H^{1,\varepsilon}$ be given by

$$\varphi^\varepsilon(z_1, z_2) = \varphi\left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}\right), \quad \psi^\varepsilon(z_1, z_2) = \psi\left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}\right) \quad \forall (z_1, z_2) \in \Omega^\varepsilon.$$

Using (3.3) we obtain that

$$\begin{aligned}
a(\bar{v}^\varepsilon(t), \Theta^\varepsilon) &= (\nu + \nu_r) \sum_{i,j=1}^{2} \int_{\Omega^\varepsilon} \frac{\partial v_i^\varepsilon(t)}{\partial z_j} \frac{\partial \varphi_i^\varepsilon(t)}{\partial z_j} dz + \alpha \sum_{i=1}^{2} \int_{\Omega^\varepsilon} \frac{\partial Z^\varepsilon(t)}{\partial z_i} \frac{\partial \psi^\varepsilon}{\partial z_i} dz \\
&= (\nu + \nu_r) \int_\Omega \sum_{i=1}^{2} \left((b_\varepsilon \cdot \nabla v_i^\varepsilon(t))(b_\varepsilon \cdot \nabla \varphi_i) + \frac{1}{(\varepsilon h^\varepsilon)^2} \frac{\partial v_i^\varepsilon(t)}{\partial y_2} \frac{\partial \varphi_i}{\partial y_2} \right) \varepsilon h^\varepsilon dy \\
&\quad + \alpha \int_\Omega \left((b_\varepsilon \cdot \nabla Z^\varepsilon(t))(b_\varepsilon \cdot \nabla \psi) + \frac{1}{(\varepsilon h^\varepsilon)^2} \frac{\partial Z^\varepsilon(t)}{\partial y_2} \frac{\partial \psi}{\partial y_2} \right) \varepsilon h^\varepsilon dy \\
&= \frac{(\nu + \nu_r)}{\varepsilon} \int_\Omega \sum_{i=1}^{2} \left((\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon(t))(\varepsilon b_\varepsilon \cdot \nabla \varphi_i) + \frac{1}{(h^\varepsilon)^2} \frac{\partial v_i^\varepsilon(t)}{\partial y_2} \frac{\partial \varphi_i}{\partial y_2} \right) h^\varepsilon dy \\
&\quad + \frac{\alpha}{\varepsilon} \int_\Omega \left((\varepsilon b_\varepsilon \cdot \nabla Z^\varepsilon(t))(\varepsilon b_\varepsilon \cdot \nabla \psi) + \frac{1}{(h^\varepsilon)^2} \frac{\partial Z^\varepsilon(t)}{\partial y_2} \frac{\partial \psi}{\partial y_2} \right) h^\varepsilon dy = \frac{1}{\varepsilon} \hat{a}(\bar{v}^\varepsilon(t), \Theta),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
B(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta^\varepsilon) &= b(v^\varepsilon(t), v^\varepsilon(t), \varphi^\varepsilon) + b_1(v^\varepsilon(t), Z^\varepsilon(t), \psi^\varepsilon) \\
&= \int_{\Omega^\varepsilon} \sum_{i,j=1}^{2} v_i^\varepsilon(t) \frac{\partial v_j^\varepsilon(t)}{\partial z_i} \varphi_j^\varepsilon dz + \sum_{i=1}^{2} \int_{\Omega^\varepsilon} v_i^\varepsilon(t) \frac{\partial Z^\varepsilon(t)}{\partial z_i} \psi^\varepsilon dz \\
&= \int_\Omega \left(\sum_{j=1}^{2} v_1^\varepsilon(t) \left(\varepsilon b_\varepsilon \cdot \nabla v_j^\varepsilon(t)\right) \varphi_j + \frac{v_2^\varepsilon(t)}{h^\varepsilon} \frac{\partial v_j^\varepsilon(t)}{\partial y_2} \varphi_j \right) h^\varepsilon dy \\
&\quad + \int_\Omega \left(\sum_{j=1}^{2} v_1^\varepsilon(t)(\varepsilon b_\varepsilon \cdot \nabla Z^\varepsilon(t)) \psi + \frac{v_2^\varepsilon(t)}{h^\varepsilon} \frac{\partial Z^\varepsilon(t)}{\partial y_2} \psi \right) h^\varepsilon dy \\
&= \hat{B}(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta),
\end{aligned} \tag{3.5}$$

$$\mathcal{R}(\bar{v}^\varepsilon(t),\Theta^\varepsilon) = -2\nu_r\int_{\Omega^\varepsilon}\left(\frac{\partial Z^\varepsilon(t)}{\partial z_2}\varphi_1^\varepsilon - \frac{\partial Z^\varepsilon(t)}{\partial z_1}\varphi_2^\varepsilon\right) + \left(\frac{\partial v_2^\varepsilon(t)}{\partial z_1} - \frac{\partial v_1^\varepsilon(t)}{\partial z_2}\right)\psi^\varepsilon dz$$
$$+4\nu_r\int_{\Omega^\varepsilon} Z^\varepsilon(t)\psi^\varepsilon dz = -2\nu_r\int_\Omega\left(\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon(t)}{\partial y_2}\varphi_1 - (\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon(t))\varphi_2\right)h^\varepsilon dy$$
$$-2\nu_r\int_\Omega\left((\varepsilon b_\varepsilon\cdot\nabla v_2^\varepsilon(t)) - \frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon(t)}{\partial y_2}\right)\psi h^\varepsilon dy + 4\nu_r\varepsilon\int_\Omega Z^\varepsilon(t)\psi\, h^\varepsilon dy$$
$$= \hat{\mathcal{R}}(\bar{v}^\varepsilon(t),\Theta). \tag{3.6}$$

Using Lemma 2.1 we have $U^\varepsilon(t,z_2) = \mathcal{U}(\frac{z_2}{\varepsilon})U_0(t) = \mathcal{U}(y_2h^\varepsilon(y_1))U_0(t)$, so

$$b_\varepsilon\cdot\nabla\mathcal{U}(y_2h^\varepsilon(y_1)) = \left(\frac{\partial}{\partial y_1} - \frac{y_2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\frac{\partial}{\partial y_2}\right)\mathcal{U}(y_2h^\varepsilon(y_1)) = \mathcal{U}'(y_2h^\varepsilon(y_1))\left(y_2\frac{\partial h^\varepsilon}{\partial y_1} - \frac{y_2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}h^\varepsilon\right)$$
$$= 0, \tag{3.7}$$

and similarly for $W^\varepsilon(t,z_2) = \mathcal{W}(\frac{z_2}{\varepsilon})W_0(t) = \mathcal{W}(y_2h^\varepsilon(y_1))W_0(t)$, so

$$b_\varepsilon\cdot\nabla\mathcal{W}(y_2h^\varepsilon(y_1)) = 0. \tag{3.8}$$

Then

$$a(\bar{\xi^\varepsilon},\Theta^\varepsilon) = (\nu+\nu_r)\int_{\Omega^\varepsilon}\nabla U^\varepsilon(z_2,t)e_1\nabla\varphi^\varepsilon dz + \alpha\int_{\Omega^\varepsilon}\nabla W^\varepsilon(z_2,t)\nabla\psi^\varepsilon dz$$
$$= (\nu+\nu_r)\int_{\Omega^\varepsilon}\sum_{i=1}^2\frac{\partial U^\varepsilon}{\partial z_i}\frac{\partial\varphi_1^\varepsilon}{\partial z_i}dz + \alpha\int_{\Omega^\varepsilon}\sum_{i=1}^2\frac{\partial W^\varepsilon}{\partial z_i}\frac{\partial\psi^\varepsilon}{\partial z_i}dz$$
$$= (\nu+\nu_r)U_0(t)\int_\Omega\left((b_\varepsilon\cdot\nabla\mathcal{U})(b_\varepsilon\cdot\nabla\varphi_1) + \frac{1}{(\varepsilon h^\varepsilon)^2}\mathcal{U}'(y_2h^\varepsilon)h^\varepsilon\frac{\partial\varphi_1}{\partial y_2}\right)\varepsilon h^\varepsilon dy$$
$$+\alpha W_0(t)\int_\Omega\left((b_\varepsilon\cdot\nabla\mathcal{W})(b_\varepsilon\cdot\nabla\psi) + \frac{1}{(\varepsilon h^\varepsilon)^2}\mathcal{W}'(y_2h^\varepsilon)h^\varepsilon\frac{\partial\psi}{\partial y_2}\right)\varepsilon h^\varepsilon dy$$
$$= \frac{(\nu+\nu_r)}{\varepsilon}U_0(t)\int_\Omega\mathcal{U}'(y_2h^\varepsilon)\frac{\partial\varphi_1}{\partial y_2}dy + \frac{\alpha}{\varepsilon}W_0(t)\int_\Omega\mathcal{W}'(y_2h^\varepsilon)\frac{\partial\psi}{\partial y_2}dy = \frac{1}{\varepsilon}\hat{a}(\bar{\xi^\varepsilon},\Theta). \tag{3.9}$$

We have also

$$B(\bar{\xi^\varepsilon},\bar{v}^\varepsilon,\Theta^\varepsilon) = b(U^\varepsilon e_1, v^\varepsilon,\varphi^\varepsilon) + b_1(U^\varepsilon e_1, Z^\varepsilon,\psi^\varepsilon)$$
$$= \int_{\Omega^\varepsilon}\sum_{j=1}^2 U^\varepsilon\frac{\partial v_j^\varepsilon}{\partial z_1}\varphi_j^\varepsilon dz + \int_{\Omega^\varepsilon}U^\varepsilon\frac{\partial Z^\varepsilon}{\partial z_1}\psi^\varepsilon dz$$
$$= U_0(t)\int_\Omega\mathcal{U}(y_2h^\varepsilon)\left(\sum_{j=1}^2(\varepsilon b_\varepsilon\cdot\nabla v_j^\varepsilon)\varphi_j + (\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon)\psi\right)h^\varepsilon dy = \hat{B}(\bar{\xi^\varepsilon},\bar{v}^\varepsilon,\Theta), \tag{3.10}$$

$$B(\bar{v}^\varepsilon,\bar{\xi^\varepsilon},\Theta^\varepsilon) = -B(\bar{v}^\varepsilon,\Theta^\varepsilon,\bar{\xi^\varepsilon}) = -b(v^\varepsilon,\varphi^\varepsilon,U^\varepsilon e_1) - b_1(v^\varepsilon,\psi^\varepsilon,W^\varepsilon)$$
$$= -\sum_{i=1}^2\int_{\Omega^\varepsilon}v_i^\varepsilon\frac{\partial\varphi_1^\varepsilon}{\partial z_i}U^\varepsilon dz - \sum_{i=1}^2\int_{\Omega^\varepsilon}v_i^\varepsilon\frac{\partial\psi^\varepsilon}{\partial z_i}W^\varepsilon dz$$
$$= -U_0(t)\int_\Omega\mathcal{U}(y_2h^\varepsilon)\left(v_1^\varepsilon(\varepsilon b_\varepsilon\cdot\nabla\varphi_1)h^\varepsilon + v_2^\varepsilon\frac{\partial\varphi_1}{\partial y_2}\right)dy$$
$$-W_0(t)\int_\Omega\mathcal{W}(y_2h^\varepsilon)\left(v_1^\varepsilon(\varepsilon b_\varepsilon\cdot\nabla\psi)h^\varepsilon + v_2^\varepsilon\frac{\partial\psi}{\partial y_2}\right)dy = \hat{B}(\bar{v}^\varepsilon,\bar{\xi^\varepsilon},\Theta), \tag{3.11}$$

$$\begin{aligned}
\mathcal{R}(\bar{\xi^\varepsilon}, \Theta^\varepsilon) &= -2\nu_r W_0(t)\int_\Omega \left(\varphi_1 \frac{1}{\varepsilon h^\varepsilon}\frac{\partial}{\partial y_2}\mathcal{W}(y_2 h^\varepsilon) - (b_\varepsilon\cdot\nabla\mathcal{W})\varphi_2\right)\varepsilon h^\varepsilon dy \\
&\quad -2\nu_r U_0(t)\int_\Omega\left(-\frac{1}{\varepsilon h^\varepsilon}\frac{\partial}{\partial y_2}\mathcal{U}(y_2h^\varepsilon)\right)\psi\varepsilon h^\varepsilon dy + 4\nu_r W_0(t)\int_\Omega \mathcal{W}(y_2h^\varepsilon)\psi\,\varepsilon h^\varepsilon dy \\
&= -2\nu_r W_0(t)\int_\Omega \mathcal{W}'(y_2h^\varepsilon)\varphi_1 h^\varepsilon dy + 2\nu_r U_0(t)\int_\Omega \mathcal{U}'(y_2h^\varepsilon)\psi h^\varepsilon dy \\
&\quad +4\nu_r W_0(t)\int_\Omega \mathcal{W}(y_2h^\varepsilon)\psi\,\varepsilon h^\varepsilon dy = \hat{\mathcal{R}}(\bar{\xi^\varepsilon}, \Theta),
\end{aligned} \tag{3.12}$$

and

$$(p^\varepsilon(t), \operatorname{div}\varphi^\varepsilon) = \int_{\Omega^\varepsilon} p^\varepsilon(t)\operatorname{div}\varphi^\varepsilon dz = \int_\Omega p^\varepsilon(t)\left((\varepsilon b_\varepsilon\cdot\nabla\varphi_1) + \frac{1}{h^\varepsilon}\frac{\partial\varphi_2}{\partial y_2}\right)h^\varepsilon dy. \tag{3.13}$$

Lemma 3.1. *Using* (3.1)-(3.2)*, the variational identity* (2.17) *written in* Ω^ε *leads to the following one in* Ω*:*

$$\begin{aligned}
&\varepsilon\int_\Omega \frac{d\bar{v}^\varepsilon}{dt}(t)\Theta^\varepsilon h^\varepsilon dy + \frac{1}{\varepsilon}\hat{a}(\bar{v}^\varepsilon(t),\Theta^\varepsilon) + \hat{B}(\bar{v}^\varepsilon(t),\bar{v}^\varepsilon(t),\Theta^\varepsilon) + \hat{\mathcal{R}}(\bar{v}^\varepsilon(t),\Theta^\varepsilon) = -\varepsilon\int_\Omega \frac{d\bar{\xi^\varepsilon}}{dt}(t)\Theta^\varepsilon h^\varepsilon dy \\
&-\frac{1}{\varepsilon}\hat{a}(\bar{\xi^\varepsilon}(t),\Theta^\varepsilon) - \hat{B}(\bar{\xi^\varepsilon}(t),\bar{v}^\varepsilon(t),\Theta^\varepsilon) - \hat{B}(\bar{v}^\varepsilon(t),\bar{\xi^\varepsilon}(t),\Theta^\varepsilon) - \hat{\mathcal{R}}(\bar{\xi^\varepsilon}(t),\Theta^\varepsilon) + \varepsilon\int_\Omega \bar{f}^\varepsilon(t)\Theta^\varepsilon h^\varepsilon dy \\
&+\int_\Omega p^\varepsilon(t)\left((\varepsilon b_\varepsilon\cdot\nabla\varphi_1^\varepsilon) + \frac{1}{h^\varepsilon}\frac{\partial\varphi_2^\varepsilon}{\partial y_2}\right)h^\varepsilon dy, \quad \forall\Theta^\varepsilon = (\varphi^\varepsilon,\psi^\varepsilon)\in V^\varepsilon\times H^{1,\varepsilon},
\end{aligned} \tag{3.14}$$

where $\hat{a}$*,* $\hat{B}$*, and* $\hat{\mathcal{R}}$*, are defined by* (3.4), (3.5), and (3.6) respectively.

Proof. Indeed, from (3.4)-(3.13), the variational identity (3.14) follows. ☐

We prove now the following uniform estimates, with respect to ε:

Proposition 3.2. *Assume that* $\varepsilon^2\bar{f}^\varepsilon$ *and* $\varepsilon\bar{v}_0^\varepsilon$ *are bounded independently of* ε *in* $(L^2((0,T)\times\Omega))^3$ *and in* $(L^2(\Omega))^3$ *respectively and* $U_0\in H^1(0,T)$*,* $W_0\in H^1(0,T)$*. There exists a constant* $C>0$ *which does not depends on* ε*, such that, for* $i=1,2$*, we have the following estimates:*

$$\|(\varepsilon b_\varepsilon\cdot\nabla v_i^\varepsilon)\|_{L^2((0,T)\times\Omega)} \le C, \qquad \|(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon)\|_{L^2((0,T)\times\Omega)} \le C, \tag{3.15}$$

$$\|\frac{\partial v_i^\varepsilon}{\partial y_2}\|_{L^2((0,T)\times\Omega)} \le C, \qquad \|\frac{\partial Z^\varepsilon}{\partial y_2}\|_{L^2((0,T)\times\Omega)} \le C, \tag{3.16}$$

$$\|\frac{\partial v_i^\varepsilon}{\partial y_1}\|_{L^2((0,T)\times\Omega)} \le \frac{C}{\varepsilon}, \qquad \|\frac{\partial Z^\varepsilon}{\partial y_1}\|_{L^2((0,T)\times\Omega)} \le \frac{C}{\varepsilon}, \tag{3.17}$$

$$\|v_i^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C, \qquad \|Z^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C. \tag{3.18}$$

Proof. Taking $\Theta^\varepsilon = \bar{v}^\varepsilon(t)$ in (3.14), and observing that $B(\bar{v}^\varepsilon(t),\bar{v}^\varepsilon(t),\bar{v}^\varepsilon(t)) = B(\bar{\xi^\varepsilon}(t),\bar{v}^\varepsilon(t),\bar{v}^\varepsilon(t)) = 0$, we obtain

$$\varepsilon\frac{d}{2dt}\int_\Omega(\bar{v}^\varepsilon(t))^2h^\varepsilon dy + \frac{(\nu+\nu_r)}{\varepsilon}\int_\Omega(\varepsilon\, b_\varepsilon\cdot\nabla v_1^\varepsilon(t))^2h^\varepsilon dy + \frac{(\nu+\nu_r)}{\varepsilon}\int_\Omega(\varepsilon\, b_\varepsilon\cdot\nabla v_2^\varepsilon(t))^2h^\varepsilon dy +$$

$$+\frac{(\nu+\nu_r)}{\varepsilon}\left(\int_\Omega\left(\frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon(t)}{\partial y_2}\right)^2 h^\varepsilon dy+\int_\Omega\left(\frac{1}{h^\varepsilon}\frac{\partial v_2^\varepsilon(t)}{\partial y_2}\right)^2 h^\varepsilon dy\right)+\frac{\alpha}{\varepsilon}\int_\Omega(\varepsilon\, b_\varepsilon\cdot\nabla Z^\varepsilon(t))^2 h^\varepsilon dy+$$
$$+\frac{\alpha}{\varepsilon}\int_\Omega\left(\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon(t)}{\partial y_2}\right)^2 h^\varepsilon dy+4\nu_r\varepsilon\int_\Omega(Z^\varepsilon(t))^2h^\varepsilon dy=2\nu_r\int_\Omega\left(\frac{\partial Z^\varepsilon(t)}{\partial y_2}v_1^\varepsilon(t)-(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon(t))v_2^\varepsilon(t)h^\varepsilon\right)dy$$
$$+2\nu_r\int_\Omega\left((\varepsilon b_\varepsilon\cdot\nabla v_2^\varepsilon(t))Z^\varepsilon(t)h^\varepsilon-\frac{\partial v_1^\varepsilon(t)}{\partial y_2}Z^\varepsilon(t)\right)dy-\frac{(\nu+\nu_r)}{\varepsilon}U_0(t)\int_\Omega\mathcal{U}'(y_2h^\varepsilon)\frac{\partial v_1^\varepsilon(t)}{\partial y_2}dy$$
$$-\frac{\alpha}{\varepsilon}W_0(t)\int_\Omega\mathcal{W}'(y_2h^\varepsilon)\frac{\partial Z^\varepsilon(t)}{\partial y_2}dy+2\nu_rW_0(t)\int_\Omega\mathcal{W}'(y_2h^\varepsilon)v_1^\varepsilon(t)h^\varepsilon dy-2\nu_rU_0(t)\int_\Omega\mathcal{U}'(y_2h^\varepsilon)Z^\varepsilon(t)h^\varepsilon dy$$
$$-4\nu_r\varepsilon W_0(t)\int_\Omega\mathcal{W}(y_2h^\varepsilon)Z^\varepsilon(t)h^\varepsilon dy+U_0(t)\int_\Omega\mathcal{U}(y_2h^\varepsilon)\left(v_1^\varepsilon(t)(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon(t))h^\varepsilon+v_2^\varepsilon(t)\frac{\partial v_1^\varepsilon(t)}{\partial y_2}\right)dy$$
$$+W_0(t)\int_\Omega\left(v_1^\varepsilon(t)(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon(t))\mathcal{W}(y_2h^\varepsilon)h^\varepsilon+v_2^\varepsilon(t)\frac{\partial Z^\varepsilon(t)}{\partial y_2}\mathcal{W}(y_2h^\varepsilon)\right)dy$$
$$-\varepsilon U_0'(t)\int_\Omega\mathcal{U}(y_2h^\varepsilon)v_1^\varepsilon(t)h^\varepsilon dy-\varepsilon W_0'(t)\int_\Omega\mathcal{W}(y_2h^\varepsilon)Z^\varepsilon(t)h^\varepsilon dy$$
$$+\varepsilon\int_\Omega g^\varepsilon(t)Z^\varepsilon(t)h^\varepsilon dy+\varepsilon\int_\Omega(f_1^\varepsilon(t)v_1^\varepsilon(t)+f_2^\varepsilon(t)v_2^\varepsilon(t))h^\varepsilon dy. \tag{3.19}$$

Now we estimate the right hand side of the above inequality (3.19).
Let λ_j for $1\le j\le 16$, which must be some strictly positive constants, such that

$$2\nu_r\left|\int_\Omega\frac{\partial Z^\varepsilon(t)}{\partial y_2}v_1^\varepsilon(t)dy\right|\le\frac{\nu_r}{\varepsilon\lambda_1}\|\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon(t)}{\partial y_2}\|^2_{L^2(\Omega)}+\varepsilon\nu_r\lambda_1h_M^2\|v_1^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.20}$$

$$2\nu_r\left|\int_\Omega(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon(t))\,v_2^\varepsilon(t)h^\varepsilon dy\right|\le\frac{\nu_r}{\varepsilon\lambda_2}\|(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon(t)\|^2_{L^2(\Omega)}$$
$$+\nu_r\varepsilon\lambda_2h_M^2\|v_2^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.21}$$

$$2\nu_r\left|\int_\Omega(\varepsilon b_\varepsilon\cdot\nabla v_2^\varepsilon(t))\,Z^\varepsilon(t)h^\varepsilon dy\right|\le\frac{\nu_r}{\varepsilon\lambda_3}\|(\varepsilon b_\varepsilon\cdot\nabla v_2^\varepsilon(t))\|^2_{L^2(\Omega)}$$
$$+\varepsilon\lambda_3\nu_rh_M^2\|Z^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.22}$$

$$2\nu_r\left|\int_\Omega\frac{\partial v_1^\varepsilon(t)}{\partial y_2}\,Z^\varepsilon(t)dy\right|\le\frac{\nu_r}{\varepsilon\lambda_4}\|\frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon(t)}{\partial y_2}\|^2_{L^2(\Omega)}+\nu_r\varepsilon\lambda_4h_M^2\|Z^\varepsilon(t)\|^2_{L^2(\Omega)} \tag{3.23}$$

$$\frac{1}{\varepsilon}\left|\int_\Omega U_0(t)\mathcal{U}'(y_2h^\varepsilon)\frac{\partial v_1^\varepsilon(t)}{\partial y_2}dy\right|\le\frac{U_0(t)^2\|\mathcal{U}'\|^2_{L^2(0,h_m)}h_ML}{2\varepsilon\lambda_5(\nu+\nu_r)}$$
$$+\frac{\lambda_5(\nu+\nu_r)}{2\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon(t)}{\partial y_2}|^2_{L^2(\Omega)}, \tag{3.24}$$

$$\frac{\alpha}{\varepsilon}\left|\int_\Omega W_0(t)\mathcal{W}'(t,y_2h^\varepsilon)\frac{\partial Z^\varepsilon(t)}{\partial y_2}h^\varepsilon dy\right|\le\frac{\alpha W_0(t)^2\|\mathcal{W}'\|^2_{L^2(0,h_m)}h_ML}{2\lambda_6\varepsilon}$$
$$+\frac{\lambda_6\alpha}{2\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon(t)}{\partial y_2}\|^2_{L^2(\Omega)} \tag{3.25}$$

$$2\nu_r \left| W_0(t)\int_\Omega \mathcal{W}'(y_2h^\varepsilon)v_1^\varepsilon(t)h^\varepsilon dy\right| \leq \frac{\nu_r W_0(t)^2\|\mathcal{W}'\|^2_{L^2(0,h_m)}L}{\varepsilon\lambda_7 h_M} + \nu_r\lambda_7\varepsilon h_M^2\|v_1^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.26}$$

$$2\nu_r \left| U_0(t)\int_\Omega \mathcal{U}'(y_2h^\varepsilon)Z^\varepsilon(t)h^\varepsilon dy\right| \leq \frac{\nu_r U_0(t)^2\|\mathcal{U}'\|^2_{L^2(0,h_m)}h_M L}{\varepsilon\lambda_8} + \nu_r\lambda_8\varepsilon\|Z^\varepsilon(t))^2\|^2_{L^2(\Omega)}, \tag{3.27}$$

$$\left| U_0(t)\int_\Omega v_1^\varepsilon(t)(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon(t))\mathcal{U}(y_2h^\varepsilon)h^\varepsilon dy\right| \leq \frac{\varepsilon U_0^2(t)\|\mathcal{U}\|^2_\infty h_M^2\lambda_9}{2}\|v_1^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon\lambda_9}\|(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon)\|^2_{L^2(\Omega)}, \tag{3.28}$$

$$\left| U_0(t)\int_\Omega v_2^\varepsilon(t)\frac{\partial v_1^\varepsilon}{\partial y_2}\mathcal{U}(y_2h^\varepsilon)dy\right| \leq \frac{1}{2\varepsilon\lambda_{10}}\|\frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon}{\partial y_2}\|^2_{L^2(\Omega)} + \frac{\varepsilon\lambda_{10}U_0^2(t)\|\mathcal{U}\|^2_\infty h_M^2}{2}\|v_2^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.29}$$

$$\left| W_0(t)\int_\Omega v_1^\varepsilon(t)(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon)\mathcal{W}(y_2h^\varepsilon)h^\varepsilon dy\right| \leq \frac{1}{2\varepsilon\lambda_{11}}\|(\varepsilon b_\varepsilon\cdot\nabla Z^\varepsilon)\|^2_{L^2(\Omega)} + \frac{\varepsilon\lambda_{11}W_0^2(t)\|\mathcal{W}\|^2_\infty h_M^2}{2}\|v_1^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.30}$$

$$\left| W_0(t)\int_\Omega v_2^\varepsilon(t)\frac{\partial Z^\varepsilon}{\partial y_2}\mathcal{W}(y_2h^\varepsilon)dy\right| \leq \frac{1}{2\varepsilon\lambda_{12}}\|\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon}{\partial y_2}\|^2_{L^2(\Omega)} + \frac{\varepsilon\lambda_{12}W_0^2(t)\|\mathcal{W}\|^2_\infty h_M^2}{2}\|v_2^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.31}$$

$$4\nu_r\varepsilon \left| W_0(t)\int_\Omega \mathcal{W}(y_2h^\varepsilon)Z^\varepsilon(t)h^\varepsilon dy\right| \leq \frac{2\nu_r\varepsilon W_0(t)^2\|\mathcal{W}\|^2_{L^2(0,h_m)}h_M L}{\lambda_{13}} + 2\varepsilon\lambda_{13}\nu_r\|Z^\varepsilon(t)\|^2_{L^2(\Omega)}, \tag{3.32}$$

$$\varepsilon \left| U_0'(t)\int_\Omega \mathcal{U}(y_2h^\varepsilon)v_1^\varepsilon(t)h^\varepsilon dy\right| \leq \frac{\varepsilon}{2}h_M^2\|v_1^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon L}{2h_M}|U_0'(t)|^2\|\mathcal{U}\|^2_{L^2(0,h_m)}, \tag{3.33}$$

$$\varepsilon|W_0'(t)\int_\Omega \mathcal{W}(y_2h^\varepsilon)Z^\varepsilon(t)h^\varepsilon dy| \leq \frac{\varepsilon}{2}h_M^2\|Z^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon L}{2h_M}|W_0'(t)|^2\|\|\mathcal{W}\|^2_{L^2(0,h_m)}. \tag{3.34}$$

Finally

$$\varepsilon\left|\int_\Omega g^\varepsilon(t)Z^\varepsilon(t)h^\varepsilon dy\right| \le \frac{h_M}{\varepsilon}\|\varepsilon^2 g^\varepsilon\|_{L^2(\Omega)}\|Z^\varepsilon\|_{L^2(\Omega)}.$$

By using Poincaré's inequality and the boundary conditions (2.12)-(2.14), we get

$$\|Z^\varepsilon\|_{L^2(\Omega)} \le \|\frac{\partial Z^\varepsilon}{\partial y_2}\|_{L^2(\Omega)} \quad \text{a.e. in } (0,T),$$

and

$$\begin{aligned}
&\varepsilon\left|\int_\Omega g^\varepsilon(t)Z^\varepsilon(t)h^\varepsilon dy\right| \le \frac{h_M}{\varepsilon}\|\varepsilon^2 g^\varepsilon\|_{L^2(\Omega)}\|\frac{\partial Z^\varepsilon}{\partial y_2}\|_{L^2(\Omega)}\\
&\le \frac{h_M^2}{\varepsilon}\|\varepsilon^2 g^\varepsilon\|_{L^2(\Omega)}\|\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon}{\partial y_2}\|_{L^2(\Omega)}\\
&\le \frac{h_M^4}{2\lambda_{14}\varepsilon}\|\varepsilon^2 g^\varepsilon\|^2_{L^2(\Omega)} + \frac{\lambda_{14}}{2\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon}{\partial y_2}\|^2_{L^2(\Omega)}. \qquad (3.35)
\end{aligned}$$

Similarly

$$\varepsilon\left|\int_\Omega f_1^\varepsilon(t)v_1^\varepsilon(t)h^\varepsilon dy\right| \le \frac{h_M^4}{2\lambda_{15}\varepsilon}\|\varepsilon^2 f_1^\varepsilon\|^2_{L^2(\Omega)} + \frac{\lambda_{15}}{2\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon}{\partial y_2}\|^2_{L^2(\Omega)}, \qquad (3.36)$$

and

$$\varepsilon\left|\int_\Omega f_2^\varepsilon(t)v_2^\varepsilon(t)h^\varepsilon dy\right| \le \frac{h_M^4}{2\lambda_{16}\varepsilon}\|\varepsilon^2 f_2^\varepsilon\|^2_{L^2(\Omega)} + \frac{\lambda_{16}}{2\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial v_2^\varepsilon}{\partial y_2}\|^2_{L^2(\Omega)}. \qquad (3.37)$$

So from (3.19) and (3.20)-(3.35) we get

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}([\overline{v}^\varepsilon(t)]^2) + \frac{c_1}{\varepsilon}\|(\varepsilon\, b_\varepsilon\cdot\nabla v_1^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{c_2}{\varepsilon}\|(\varepsilon\, b_\varepsilon\cdot\nabla v_2^\varepsilon(t)\|^2_{L^2(\Omega)}\\
&+\frac{c_3}{\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial Z^\varepsilon(t)}{\partial y_2}\|^2_{L^2(\Omega)} + \frac{c_4}{\varepsilon}\|\frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon(t)}{\partial y_2}\|^2_{L^2(\Omega)} + \frac{c_5}{\varepsilon}h_m\|\frac{1}{h^\varepsilon}\frac{\partial v_2^\varepsilon(t)}{\partial y_2}\|^2_{L^2(\Omega)}\\
&+\frac{c_6}{\varepsilon}\|(\varepsilon\, b_\varepsilon\cdot\nabla Z^\varepsilon(t)\|^2_{L^2(\Omega)} + \nu_r\varepsilon c_7\|Z^\varepsilon(t)\|^2_{L^2(\Omega)} \le \varepsilon c_8(t)[\overline{v}^\varepsilon(t)]^2 + \frac{c_9(t)}{\varepsilon} \qquad (3.38)
\end{aligned}$$

where

$c_1 = (\nu+\nu_r)h_m - \dfrac{1}{2\lambda_9}, \qquad c_2 = (\nu+\nu_r)h_m - \dfrac{\nu_r}{\lambda_3},$

$c_3 = \alpha h_m - \dfrac{\nu_r}{\lambda_1} - \dfrac{\lambda_6\alpha}{2} - \dfrac{1}{2\lambda_{12}} - \dfrac{\lambda_{14}}{2}, \quad c_4 = (\nu+\nu_r)h_m - \dfrac{\lambda_5(\nu+\nu_r)^2}{2} - \dfrac{\nu_r}{\lambda_4} - \dfrac{1}{2\lambda_{10}} - \dfrac{\lambda_{15}}{2},$

$c_5 = (\nu+\nu_r)h_m - \dfrac{\lambda_{16}}{2}, \quad c_6 = \alpha h_m - \dfrac{1}{2\lambda_{11}} - \dfrac{\nu_r}{\lambda_2}, \quad c_7 = 4h_m - \lambda_8 - 2\lambda_{13},$

$c_8(t) = \max\{A(t), B(t), h_M^2(1+\lambda_3+\lambda_4)\}$

with

$$A(t) = h_M^2\left(1 + \nu_r\lambda_1 + \nu_r\lambda_7 + \frac{\lambda_9 U_0^2(t)\|\mathcal{U}\|^2_\infty + \lambda_{11}W_0^2(t)\|\mathcal{W}\|^2_\infty}{2}\right)$$

$$B(t) = h_M^2 \left(\frac{1}{2} + \nu_r \lambda_2 + \frac{\lambda_{10} U_0^2(t) \|\mathcal{U}\|_\infty^2}{2} + \frac{\lambda_{12} W_0^2(t) \|\mathcal{W}\|_\infty^2}{2} \right)$$

and

$$\begin{aligned}\frac{c_9(t)}{\varepsilon} = & \frac{U_0^2(t)\|\mathcal{U}'\|^2_{L^2(0,h_m)} h_M L}{2\varepsilon\lambda_5} + \frac{\nu_r U_0^2(t)\|\mathcal{U}\|^2_{L^2(0,h_m)} h_M L}{\varepsilon\lambda_8} + \frac{\alpha W_0^2(t)\|\mathcal{W}'\|^2_{L^2(0,h_m)} h_M L}{2\varepsilon\lambda_6} \\ & + \frac{\nu_r W_0^2(t)\|\mathcal{W}'\|^2_{L^2(0,h_m)} L}{\varepsilon\lambda_7 h_M} + \frac{2\nu_r \varepsilon W_0(t)^2 \|\mathcal{W}\|^2_{L^2(0,h_m)} h_M L}{\lambda_{13}} + \frac{\varepsilon L}{2h_M} |U_0'(t)|^2 \|\mathcal{U}\|^2_{L^2(0,h_m)} \\ & + \frac{\varepsilon L}{2h_M} |W_0'(t)|^2 \|\mathcal{W}\|^2_{L^2(0,h_m)} + \frac{h_M^4}{2\varepsilon} \left(\frac{1}{\lambda_{15}} \|\varepsilon^2 f_1^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{1}{\lambda_{16}} \|\varepsilon^2 f_2^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{1}{\lambda_{14}} \|\varepsilon^2 g^\varepsilon(t)\|^2_{L^2(\Omega)} \right).\end{aligned}$$

Each c_i for $i = 1, \cdots, 6$ must be strictly positive, which is possible for example with

$$\lambda_1 = \lambda_2 = \frac{4\nu_r}{\alpha h_m}, \qquad \lambda_3 = \lambda_4 = \frac{1}{h_m}, \qquad \lambda_5 = \frac{\nu h_m}{2(\nu+\nu_r)^2},$$
$$\lambda_6 = \lambda_8 = \lambda_{13} = \frac{h_m}{2}, \qquad \lambda_9 = \frac{4}{(\nu+\nu_r)h_m}, \qquad \lambda_{10} = \frac{2}{\nu h_m}, \qquad \lambda_{11} = \lambda_{12} = \frac{2}{\alpha h_m},$$
$$\lambda_{14} = \frac{\alpha h_m}{8}, \qquad \lambda_{15} = \frac{\nu h_m}{4}, \qquad \lambda_{16} = \frac{(\nu+\nu_r)h_m}{2}.$$

Note that λ_7 remains arbitrary and can be taken as $\lambda_7 = 1$. So from (3.38) we get

$$\frac{\varepsilon^2}{2} \frac{d}{dt}([\bar{v}^\varepsilon(t)]^2) \le \varepsilon^2 c_7(t) [\bar{v}^\varepsilon(t)]^2 + c_8(t).$$

As U_0 and W_0 belong to $H^1(0,T)$, then c_8 is bounded in $L^1(0,T)$ independently of ε and by Grönwall's lemma we deduce that there exists a constant C independent of ε such that

$$\varepsilon^2 [\bar{v}^\varepsilon(t)]^2 \le C \quad \forall t \in [0,T]. \tag{3.39}$$

Now we integrate the inequality (3.38) over the time interval $(0,s)$ for $0 < s \le T$, we deduce

$$\begin{aligned} & \frac{\varepsilon^2}{2} [\bar{v}^\varepsilon(s)]^2 + C_1 \int_0^s \|(\varepsilon\, b_\varepsilon \cdot \nabla v_1^\varepsilon(t)\|^2_{L^2(\Omega)} + \|(\varepsilon\, b_\varepsilon \cdot \nabla v_2^\varepsilon(t)\|^2_{L^2(\Omega)} + + \|(\varepsilon\, b_\varepsilon \cdot \nabla Z^\varepsilon(t)\|^2_{L^2(\Omega)} dt \\ & + C_2 \int_0^s \| \frac{1}{h^\varepsilon} \frac{\partial Z^\varepsilon(t)}{\partial y_2} \|^2_{L^2(\Omega)} + \| \frac{1}{h^\varepsilon} \frac{\partial v_1^\varepsilon(t)}{\partial y_2} \|^2_{L^2(\Omega)} + \| \frac{1}{h^\varepsilon} \frac{\partial v_2^\varepsilon(t)}{\partial y_2} \|^2_{L^2(\Omega)} dt \\ & + \varepsilon^2 \nu_r c_6 \int_0^s \|(Z^\varepsilon(t)\|^2_{L^2(\Omega)} dt \le \varepsilon^2 \int_0^s c_7(t) [\bar{v}^\varepsilon(t)]^2 dt + \int_0^s c_8(t) dt + \frac{\varepsilon^2}{2} [\bar{v}^\varepsilon(0)]^2, \end{aligned} \tag{3.40}$$

where $C_1 = \min\{c_1, c_2, c_6\}$, $C_2 = \min\{c_3, c_4, c_5\}$, are two constants independent of ε. Observing that $c_7 \in L^\infty(0,T)$ and

$$\int_0^T \int_\Omega \left(\frac{1}{h^\varepsilon} \frac{\partial v_i^\varepsilon(t)}{\partial y_2} \right)^2 dy dt \ge \frac{1}{h_M^2} \| \frac{\partial v_i^\varepsilon}{\partial y_2} \|^2_{L^2((0,T)\times\Omega)}$$

we deduce (3.15) and (3.16) from (3.39). Moreover, from (3.3)

$$b_\varepsilon \cdot \nabla = \frac{\partial}{\partial y_1} - \frac{y_2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y_1} \frac{\partial}{\partial y_2} \quad \text{with} \quad |\frac{\partial h^\varepsilon}{\partial y_1}| = |\frac{\partial h}{\partial y_1} + \frac{1}{\varepsilon} \frac{\partial h}{\partial \eta_1}| \le \frac{C}{\varepsilon}. \tag{3.41}$$

Thus we have

$$\int_0^T \int_\Omega \left(\varepsilon \frac{\partial v_i^\varepsilon(t)}{\partial y_1}\right)^2 dydt = \int_0^T \int_\Omega \left((\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon(t)) + \frac{y_2 \varepsilon}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\frac{\partial v_i^\varepsilon(t)}{\partial y_2}\right)^2 dydt$$
$$\leq 2\|(\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon)\|^2_{L^2((0,T)\times\Omega)} + 2\frac{C}{h_m}\|\frac{\partial v_i^\varepsilon}{\partial y_2}\|^2_{L^2((0,T)\times\Omega)}$$

and a similar estimate holds for Z^ε. Finally with (3.15) and (3.16) we deduce (3.17). Next, using again the boundary conditions (2.12)-(2.14) and Poincaré's inequality, we get

$$\|v_i^\varepsilon\|^2_{L^2((0,T)\times\Omega)} \leq \int_0^T \int_\Omega \|\frac{\partial v_i^\varepsilon}{\partial y_2}\|^2 dydt = \|\frac{\partial v_i^\varepsilon}{\partial y_2}\|^2_{L^2((0,T)\times\Omega)}$$

and we deduce (3.18) from (3.16). □

Proposition 3.3. *Assume that the proposition* 3.2 *holds. Then there exists a constant* $C > 0$ *which does not depends on* ε*, such that we have*

$$\varepsilon^2\|p^\varepsilon\|_{H^{-1}(0,T;L^2(\Omega))} \leq C. \tag{3.42}$$

Proof. Let $\varphi \in \mathcal{D}(0,T) \times \mathcal{D}(\Omega)$, then choose $\Theta = (0, \varphi(t), 0)$ as a test-function in (3.14) and multiply the two sides by ε: we obtain

$$\varepsilon \int_0^T \int_\Omega p^\varepsilon \frac{\partial \varphi}{\partial y_2} dydt = -\varepsilon^2 \int_0^T \int_\Omega v_2^\varepsilon \frac{\partial \varphi}{\partial t} h^\varepsilon dydt$$
$$+(\nu+\nu_r)\sum_{i=1}^2 \int_0^T \int_\Omega \left(h^\varepsilon(\varepsilon b_\varepsilon \cdot \nabla v_2^\varepsilon)(\varepsilon b_\varepsilon \cdot \nabla \varphi) + \frac{1}{h^\varepsilon}\frac{\partial v_2^\varepsilon}{\partial y_2}\frac{\partial \varphi}{\partial y_2}\right) dydt$$
$$+ \int_0^T \int_\Omega \left(\sum_{i=1}^2 \varepsilon v_1^\varepsilon(\varepsilon b_\varepsilon \cdot \nabla v_2^\varepsilon)\varphi h^\varepsilon + \varepsilon v_2^\varepsilon \frac{\partial v_2^\varepsilon}{\partial y_2}\varphi\right) dydt + 2\nu_r \int_0^T \int_\Omega (\varepsilon b_\varepsilon \cdot \nabla Z^\varepsilon)\varphi \varepsilon h^\varepsilon dydt$$
$$+ \int_0^T \int_\Omega U_\varepsilon(\varepsilon b_\varepsilon \cdot \nabla v_2^\varepsilon)\varphi\varepsilon h^\varepsilon dydt - \int_0^T \int_\Omega \varepsilon^2 f_2^\varepsilon \varphi h^\varepsilon dydt, \tag{3.43}$$

with $U_\varepsilon(t,y) = U_0(t)\mathcal{U}(y_2 h^\varepsilon(y_1))$ for all $(t, y_1, y_2) \in [0,T] \times \Omega$. Using (3.15)-(3.18), we get

$$|\int_0^T \int_\Omega p^\varepsilon \frac{\partial \varphi}{\partial y_2} dydt| \leq \frac{C}{\varepsilon}\|\varphi\|_{H^1(0,T,H_0^1(\Omega))} \quad \forall \varphi \in \mathcal{D}(0,T) \times \mathcal{D}(\Omega). \tag{3.44}$$

Now let $\phi \in \mathcal{D}(0,T) \times \mathcal{D}(\Omega)$ and choose $\Theta = (\frac{\phi}{h^\varepsilon}, 0, 0)$ as a test-function in (3.14), then multiply the two sides by ε: we obtain

$$\varepsilon^2 \int_0^T \int_\Omega p^\varepsilon \left(\frac{\partial \phi}{\partial y_1} - \frac{\partial}{\partial y_2}\left(y_2 \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\phi\right)\right) dydt = -\varepsilon^2 \int_0^T \int_\Omega v_1^\varepsilon \frac{\partial \phi}{\partial t} dydt$$
$$+(\nu+\nu_r)\int_0^T \int_\Omega \varepsilon(\varepsilon b_\varepsilon \cdot \nabla v_1^\varepsilon)\left(\frac{\partial \phi}{\partial y_1} - \frac{\partial h^\varepsilon}{\partial y_1}\frac{1}{h^\varepsilon}\phi - y_2 \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\frac{\partial \phi}{\partial y_2}\right) dydt$$
$$+(\nu+\nu_r)\int_0^T \int_\Omega \frac{1}{(h^\varepsilon)^2}\frac{\partial v_1^\varepsilon}{\partial y_2}\frac{\partial \phi}{\partial y_2} dydt + \int_0^T \int_\Omega \left(\varepsilon v_1^\varepsilon(\varepsilon b_\varepsilon \cdot \nabla v_1^\varepsilon)\phi + \frac{\varepsilon}{h^\varepsilon} v_2^\varepsilon \frac{\partial v_1^\varepsilon}{\partial y_2}\phi\right) dydt$$

$$+(\nu+\nu_r)\int_0^T\int_\Omega \frac{1}{(h^\varepsilon)^2}\frac{\partial U_\varepsilon}{\partial y_2}\frac{\partial \phi}{\partial y_2}dydt - 2\nu_r\int_0^T\int_\Omega \frac{\varepsilon}{h^\varepsilon}\frac{\partial Z^\varepsilon}{\partial y_2}\phi dydt + \int_0^T\int_\Omega \varepsilon U_\varepsilon(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon)\phi dydt$$
$$-\int_0^T\int_\Omega \varepsilon^2 v_1^\varepsilon\left(\frac{\partial\phi}{\partial y_1}-\frac{\partial h^\varepsilon}{\partial y_1}\frac{1}{h^\varepsilon}\phi - y_2\frac{\partial h^\varepsilon}{\partial y_1}\frac{1}{h^\varepsilon}\frac{\partial\phi}{\partial y_2}\right)U_\varepsilon dydt - \int_0^T\int_\Omega \frac{\varepsilon}{h^\varepsilon}v_2^\varepsilon\frac{\partial\phi}{\partial y_2}U_\varepsilon dydt$$
$$-2\varepsilon\nu_r\int_0^T\int_\Omega \frac{1}{h^\varepsilon}\frac{\partial W_\varepsilon}{\partial y_2}\phi dydt - \int_0^T\int_\Omega\left(f_1^\varepsilon\phi - \frac{\partial U^\varepsilon}{\partial t}\phi\right)\varepsilon^2 dydt, \tag{3.45}$$

where $W_\varepsilon(t,y)=W_0(t)\mathcal{W}(y_2h^\varepsilon(y_1))$ for all $(t,y_1,y_2)\in[0,T]\times\Omega$.

Using the estimates (3.15)-(3.18) and (3.41), we infer that

$$|\int_0^T\int_\Omega p^\varepsilon\left(\frac{\partial\phi}{\partial y_1}-\frac{\partial}{\partial y_2}\left(y_2\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\phi\right)\right)dydt| \le \frac{C}{\varepsilon^2}\|\phi\|_{H^1(0,T;H^1(\Omega))}.$$

By choosing now $\varphi = y_2\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\phi$ in (3.43), we get

$$\frac{\partial\varphi}{\partial t} = y_2\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\frac{\partial\phi}{\partial t},\quad \frac{\partial\varphi}{\partial y_2} = \frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\left(\phi + y_2\frac{\partial\phi}{\partial y_2}\right)$$

and

$$b_\varepsilon\cdot\nabla\varphi = -y_2\frac{1}{(h^\varepsilon)^2}\left(\frac{\partial h^\varepsilon}{\partial y_1}\right)^2\left(2\phi + y_2\frac{\partial\phi}{\partial y_2}\right) + y_2\frac{1}{h^\varepsilon}\left(\frac{\partial^2 h^\varepsilon}{\partial y_1^2}\phi + \frac{\partial h^\varepsilon}{\partial y_1}\frac{\partial\phi}{\partial y_1}\right).$$

Hence

$$\|\varphi\|_{L^\infty(0,T;L^4(\Omega))}\le\frac{C}{\varepsilon}\|\phi\|_{H^1(0,T;H^1(\Omega))},\quad \|\frac{\partial\varphi}{\partial t}\|_{L^2((0,T)\times\Omega)}\le\frac{C}{\varepsilon}\|\phi\|_{H^1(0,T;H^1(\Omega))},$$
$$\|\frac{\partial\varphi}{\partial y_2}\|_{L^2((0,T)\times\Omega)}\le\frac{C}{\varepsilon}\|\phi\|_{H^1(0,T;H^1(\Omega))},\quad \|\varepsilon b_\varepsilon\cdot\nabla\varphi\|_{L^2((0,T)\times\Omega)}\le\frac{C}{\varepsilon}\|\phi\|_{H^1(0,T;H^1(\Omega))}$$

and with (3.43)

$$|\int_0^T\int_\Omega p^\varepsilon\frac{\partial}{\partial y_2}\left(y_2\frac{1}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\phi\right)dydt|\le\frac{C}{\varepsilon^2}\|\phi\|_{H^1(0,T;H^1(\Omega))}.$$

It follows that

$$|\int_0^T\int_\Omega p^\varepsilon\frac{\partial\phi}{\partial y_1}dydt|\le\frac{C}{\varepsilon^2}\|\phi\|_{H^1(0,T;H^1(\Omega))}\quad \forall\phi\in\mathcal{D}(0,T)\times\mathcal{D}(\Omega). \tag{3.46}$$

By density of $\mathcal{D}(0,T)\times\mathcal{D}(\Omega)$ into $H_0^1(0,T;H_0^1(\Omega))$ we get from (3.44)-(3.46)

$$\|\frac{\partial p^\varepsilon}{\partial y_2}\|_{H^{-1}(0,T;H^{-1}(\Omega))}\le\frac{C}{\varepsilon},\quad \|\frac{\partial p^\varepsilon}{\partial y_1}\|_{H^{-1}(0,T;H^{-1}(\Omega))}\le\frac{C}{\varepsilon^2}. \tag{3.47}$$

Finally we can deduce [28] that $\varepsilon^2 p^\varepsilon$ remains in a bounded subset of $H^{-1}(0,T;L^2(\Omega))$.
□

4. Two-scale convergence properties. Since our unknown functions depend on the time variable, we are not in the classical framework of two-scale convergence as it has been introduced by G. Allaire in [1] or G. Nguetseng in [25]. Nevertheless this technique can be easily adpated to a time-dependent framework (see for instance [24, 18, 17, 29]). For the convenience of the reader we will provide a complete proof a the generalization of [1] that will be used later for the study of the sequences $(v^\varepsilon)_{\varepsilon>0}$, $(Z^\varepsilon)_{\varepsilon>0}$ and $(p^\varepsilon)_{\varepsilon>0}$.

Let us recall the following usual notations: $Y = [0,1]^2$, $\mathcal{C}^\infty_\sharp(Y)$ is the space of infinitely differentiable functions in $\mathbb{R}^2$ that are Y-periodic and

$$L^2_\sharp(Y) = \overline{\mathcal{C}^\infty_\sharp(Y)}^{L^2(Y)}, \quad H^1_\sharp(Y) = \overline{\mathcal{C}^\infty_\sharp(Y)}^{H^1(Y)}.$$

Remark 4.1. *The space $L^2_\sharp(Y)$ coincides with the space of functions of $L^2(Y)$ extended by Y-periodicity to $\mathbb{R}^2$.*

We extend the definition of the two-scale convergence as follows

Definition 4.1. *A sequence $(w^\varepsilon)_{\varepsilon>0}$ of $L^2\big((0,T)\times\Omega\big)$ (resp. in $H^{-1}\big(0,T;L^2(\Omega)\big)$) two-scale converges to $w^0 \in L^2\big(0,T;L^2(\Omega\times Y)\big)$ (resp. $w^0 \in H^{-1}\big(0,T;L^2(\Omega\times Y)\big)$) if and only if*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega w^\varepsilon(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt = \int_0^T\int_{\Omega\times Y} w^0(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt$$

for all $\theta \in \mathcal{D}(0,T)$, for all $\varphi \in \mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(Y)\big)$. In such a case we will denote $w^\varepsilon \twoheadrightarrow w^0$.

Then we obtain

Theorem 4.2. *Let $(w^\varepsilon)_{\varepsilon>0}$ be a bounded sequence of $L^2\big((0,T)\times\Omega\big)$ (resp. in $H^{-1}\big(0,T;L^2(\Omega)\big)$). There exists $w^0 \in L^2\big(0,T;L^2(\Omega\times Y)\big)$ (resp. $w^0 \in H^{-1}\big(0,T;L^2(\Omega\times Y)\big)$) such that, possibly extracting a subsequence still denoted $(w^\varepsilon)_{\varepsilon>0}$, we have*

$$w^\varepsilon \twoheadrightarrow w^0.$$

Proof. The proof is similar to the proof of Theorem 1.2 in [1]. Let us assume first that $(w^\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $L^2\big((0,T)\times\Omega\big)$. In our time-dependent framework we consider test-functions $\psi \in \mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$. Furthermore, For any $\psi \in \mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ and for any fixed $\varepsilon > 0$, the mapping $(t,y)\mapsto \psi^\varepsilon(t,y) = \psi\left(t,y,\frac{y}{\varepsilon}\right)$ is mesurable on $(0,T)\times\Omega$ and satisfies

$$\|\psi^\varepsilon\|_{L^2((0,T)\times\Omega)} = \left(\int_0^T\int_\Omega\left(\psi\left(t,y,\frac{y}{\varepsilon}\right)\right)^2 dydt\right)^{1/2} \le \sqrt{T|\Omega|}\,\|\psi\|_{\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y)))}.$$

Hence we can define $\Lambda_\varepsilon \in \Big(\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)\Big)'$ by

$$\Lambda_\varepsilon(\psi) = \int_0^T\int_\Omega w^\varepsilon(t,y)\psi\left(t,y,\frac{y}{\varepsilon}\right)\,dydt \quad \forall\psi\in\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big).$$

Since $(w^\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $L^2\big((0,T)\times\Omega\big)$, we infer with Cauchy-Schwarz's inequality that there exists a real number $C > 0$, independent of ε, such that

$$\begin{aligned}\big|\Lambda_\varepsilon(\psi)\big| &\le \|w^\varepsilon\|_{L^2((0,T)\times\Omega)}\|\psi^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C\|\psi^\varepsilon\|_{L^2((0,T)\times\Omega)}\\ &\le C\sqrt{T|\Omega|}\,\|\psi\|_{\mathcal{C}([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y)))}\end{aligned}\tag{4.1}$$

for all $\psi \in \mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ and the sequence $(\Lambda_\varepsilon)_{\varepsilon>0}$ is bounded in $\Big(\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)\Big)'$. Reminding that $\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ is a separable Banach space, we infer that there exists $\Lambda_0 \in \Big(\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)\Big)'$ such that, possibly extracting a subsequence still denoted $(\Lambda_\varepsilon)_{\varepsilon>0}$,

$$(\Lambda_\varepsilon) \rightharpoonup \Lambda_0 \quad \text{weak * in } \Big(\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)\Big)'$$

i.e.

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega w^\varepsilon(t,y)\psi\left(t,y,\frac{y}{\varepsilon}\right)dydt = \Lambda_0(\psi) \quad \forall\psi\in\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big).$$

Observing that, for all $t\in[0,T]$, $\psi^2(t,\cdot,\cdot)\in L^1\big(\Omega;\mathcal{C}_\sharp(Y)\big)$, we can also use Lemma 5.2 of [1], which yields

$$\lim_{\varepsilon\to 0}\int_\Omega\left(\psi\left(t,y,\frac{y}{\varepsilon}\right)\right)^2 dy = \int_{\Omega\times Y}\big(\psi(t,y,\eta)\big)^2 d\eta dy \quad \forall t\in[0,T].$$

Then, using Lebesgue's convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\left(\psi\left(t,y,\frac{y}{\varepsilon}\right)\right)^2 dydt = \int_0^T\int_{\Omega\times Y}\big(\psi(t,y,\eta)\big)^2 d\eta dydt. \tag{4.2}$$

With (4.1) and (4.2) we get

$$\big|\Lambda_0(\psi)\big| \le C\|\psi\|_{L^2(0,T;L^2(\Omega\times Y))} \quad \forall\psi\in\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big).$$

It follows that Λ_0 can be extended to $\Big(L^2\big(0,T;L^2(\Omega\times Y)\big)\Big)'$ and with Riesz's representation theorem we infer that there exists $w^0\in L^2\big(0,T;L^2(\Omega\times Y)\big)$ such that

$$\Lambda_0(\psi) = \int_0^T\int_{\Omega\times Y} w^0(t,y,\eta)\psi(t,y,\eta)\,d\eta dydt \quad \forall\psi\in L^2\big(0,T;L^2(\Omega\times Y)\big)$$

which allows us to conclude for the first part of the theorem.

Let us assume now that $(w^\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $H^{-1}\big(0,T;L^2(\Omega)\big)$ and let

$$\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big) = \big\{\psi\in\mathcal{C}^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big);\psi(0,y,\eta)=\psi(T,y,\eta)=0\ \forall(y,\eta)\in\overline{\Omega}\times Y\big\}.$$

It is a closed subspace of $\mathcal{C}^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ for the usual norm of $\mathcal{C}^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ and for any $\psi\in\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$, we have

$$\|\psi^\varepsilon\|_{H^1(0,T;L^2(\Omega))} = \left(\int_0^T\int_\Omega\left(\psi\left(t,y,\frac{y}{\varepsilon}\right)\right)^2 dydt + \int_0^T\int_\Omega\left(\frac{\partial\psi}{\partial t}\left(t,y,\frac{y}{\varepsilon}\right)\right)^2 dydt\right)^{1/2}$$
$$\le \sqrt{T|\Omega|}\|\psi\|_{\mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y)))}.$$

Furthermore, we may now define $\Lambda_\varepsilon\in\Big(\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)\Big)'$ by

$$\Lambda_\varepsilon(\psi) = \int_0^T\int_\Omega w^\varepsilon(t,y)\psi\left(t,y,\frac{y}{\varepsilon}\right)dydt \quad \forall\psi\in\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big).$$

Since $(w^\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $H^{-1}\big(0,T;L^2(\Omega)\big)$, we infer that there exists a real number $C'>0$, independent of ε, such that

$$\big|\Lambda_\varepsilon(\psi)\big| \le \|w^\varepsilon\|_{H^{-1}(0,T;L^2(\Omega))}\|\psi^\varepsilon\|_{H^1(0,T;L^2(\Omega))} \le C'\|\psi^\varepsilon\|_{H^1(0,T;L^2(\Omega))} \le C'\sqrt{T|\Omega|}\|\psi\|_{\mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y)))}$$

for all $\psi\in\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ and the sequence $(\Lambda_\varepsilon)_{\varepsilon>0}$ is bounded in $\Big(\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)\Big)'$. Since $\mathcal{C}^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ is a separable Banach space, we can conclude in the same way as previously. $\square$

Remark 4.2. *We may observe that this proof shows that we can choose test-functions in* $\mathcal{C}\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$ *(resp. in* $\mathcal{C}_0^1\big([0,T];\mathcal{C}(\overline{\Omega};\mathcal{C}_\sharp(Y))\big)$*) instead of* $\mathcal{D}(0,T)\times\mathcal{D}\big(\Omega;\mathcal{C}_\sharp^\infty(Y)\big)$.

Then the convergence results for the velocity, the micro-rotation and the pressure are given in the following three propositions.

Proposition 4.3. **(Two-scale limit of the velocity)** *Under the assumptions of Proposition 3.2, there exist* $v^0\in\Big(L^2\big(0,T;L^2(\Omega;H^1_\sharp(Y))\big)\Big)^2$ *such that* $\dfrac{\partial v^0}{\partial y_2}\in\Big(L^2\big(0,T;L^2(\Omega\times Y)\big)\Big)^2$ *and* $v^1\in\Big(L^2\big(0,T;L^2\big(\Omega\times(0,1);H^1_\sharp(0,1)_{/\mathbb{R}}\big)\big)\Big)^2$ *such that, possibly extracting a subsequence still denoted* $(v^\varepsilon)_{\varepsilon>0}$*, we have for* $i=1,2$*:*

$$v_i^\varepsilon \twoheadrightarrow v_i^0,\quad \frac{\partial v_i^\varepsilon}{\partial y_2}\twoheadrightarrow\frac{\partial v_i^0}{\partial y_2}+\frac{\partial v_i^1}{\partial\eta_2}, \tag{4.3}$$

and

$$\varepsilon\frac{\partial v_i^\varepsilon}{\partial y_1}\twoheadrightarrow\frac{\partial v_i^0}{\partial\eta_1}. \tag{4.4}$$

Furthermore v^0 *does not depend on* η_2*,* v^0 *is divergence free in the following sense*

$$h(y_1,\eta_1)\frac{\partial v_1^0}{\partial\eta_1}-y_2\frac{\partial h}{\partial\eta_1}(y_1,\eta_1)\frac{\partial v_1^0}{\partial y_2}+\frac{\partial v_2^0}{\partial y_2}=0\quad\text{in } (0,T)\times\Omega\times(0,1), \tag{4.5}$$

and

$$v^0=0\quad\text{on } (0,T)\times\Gamma_0\times(0,1),\Gamma_0=(0,L)\times\{0\}, \tag{4.6}$$

$$-v_1^0\frac{\partial h}{\partial\eta_1}(y_1,\eta_1)+v_2^0=0\quad\text{on } (0,T)\times\Gamma_1\times(0,1),\Gamma_1=(0,L)\times\{1\}. \tag{4.7}$$

Proof. The first part of the result is a direct consequence of the previous theorem and is obtained by using the same techniques as in Proposition 1.14 in [1].

Indeed, from Proposition 3.2 we know that $(v_i^\varepsilon)_{\varepsilon>0}$, $\left(\dfrac{\partial v_i^\varepsilon}{\partial y_2}\right)_{\varepsilon>0}$ and $\left(\varepsilon\dfrac{\partial v_i^\varepsilon}{\partial y_1}\right)_{\varepsilon>0}$ are bounded in $L^2\big((0,T)\times\Omega\big)$. It follows that, possibly extracting a subsequence, they two-scale converge to v_i^0, ξ_i^0 and ξ_i^1 respectively, i.e.

$$\lim_{\varepsilon\to0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt=\int_0^T\int_{\Omega\times Y}v_i^0(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt \tag{4.8}$$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\partial v_i^\varepsilon}{\partial y_2}(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt \\
&= -\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\left(\frac{\partial\varphi}{\partial y_2}\left(y,\frac{y}{\varepsilon}\right)+\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial\eta_2}\left(y,\frac{y}{\varepsilon}\right)\right)\theta(t)\,dydt \\
&= \int_0^T\int_{\Omega\times Y}\xi_i^0(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varepsilon\frac{\partial v_i^\varepsilon}{\partial y_1}(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt \\
&= -\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\left(\varepsilon\frac{\partial\varphi}{\partial y_1}\left(y,\frac{y}{\varepsilon}\right)+\frac{\partial\varphi}{\partial\eta_1}\left(y,\frac{y}{\varepsilon}\right)\right)\theta(t)\,dydt \\
&= \int_0^T\int_{\Omega\times Y}\xi_i^1(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt
\end{aligned}
\tag{4.10}
$$

for all $\theta\in\mathcal{D}(0,T)$, $\varphi\in\mathcal{D}\big(\Omega;\mathcal{C}_\sharp^\infty(Y)\big)$. From (4.10) and (4.8) we obtain

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\left(\varepsilon\frac{\partial\varphi}{\partial y_1}\left(y,\frac{y}{\varepsilon}\right)+\frac{\partial\varphi}{\partial\eta_1}\left(y,\frac{y}{\varepsilon}\right)\right)\theta(t)\,dydt \\
&= -\int_0^T\int_{\Omega\times Y} v_i^0(t,y,\eta)\frac{\partial\varphi}{\partial\eta_1}(y,\eta)\theta(t)\,d\eta dydt \\
&= \int_0^T\int_{\Omega\times Y}\xi_i^1(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt
\end{aligned}
\tag{4.11}
$$

for all $\theta\in\mathcal{D}(0,T)$, $\varphi\in\mathcal{D}\big(\Omega;\mathcal{C}_\sharp^\infty(Y)\big)$, which implies that $\xi_i^1=\dfrac{\partial v_i^0}{\partial\eta_1}\in L^2\big(0,T;L^2(\Omega\times Y)\big)$. Thus (4.4) holds.

Similarly, by multiplying (4.9) by ε and taking into account (4.8) we get

$$
\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\frac{\partial\varphi}{\partial\eta_2}\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt = 0 = \int_0^T\int_{\Omega\times Y} v_i^0(t,y,\eta)\frac{\partial\varphi}{\partial\eta_2}(y,\eta)\theta(t)\,d\eta dydt
$$

and thus v_i^0 does not depend on η_2. Moreover, by choosing φ independent of η (i.e $\varphi\in\mathcal{D}(\Omega)$) in (4.11), we get

$$
\int_0^T\int_{\Omega\times(0,1)}\frac{\partial v_i^0}{\partial\eta_1}(t,y,\eta_1)\varphi(y)\theta(t)\,d\eta_1 dydt = 0 = \int_0^T\int_\Omega\big(v_i^0(t,y,1)-v_i^0(t,y,0)\big)\varphi(y)\theta(t)\,dydt
$$

and $v_i^0\in L^2\big(0,T;L^2(\Omega;H_\sharp^1(Y))\big)$.

Next, by choosing $\varphi\in\mathcal{D}\big(\Omega\times(0,1)\big)$ (i.e. φ does not depend on η_2), we obtain now

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\frac{\partial\varphi}{\partial y_2}\left(y,\frac{y_1}{\varepsilon}\right)\theta(t)\,dydt = \int_0^T\int_{\Omega\times Y} v_i^0(t,y,\eta_1)\frac{\partial\varphi}{\partial y_2}(y,\eta_1)\theta(t)\,d\eta dydt \\
&= -\int_0^T\int_{\Omega\times Y}\xi_i^0(t,y,\eta)\varphi(y,\eta_1)\theta(t)\,d\eta dydt.
\end{aligned}
$$

Hence

$$
\int_0^T\int_{\Omega\times Y}\left(-\frac{\partial v_i^0}{\partial y_2}(t,y,\eta_1)+\xi_i^0(t,y,\eta)\right)\varphi(y_1,y_2,\eta_1)\theta(t)\,d\eta dydt = 0.
$$

It follows that there exists $v_i^1 \in L^2\big(0,T; L^2\big(\Omega \times (0,1); H^1_\sharp(0,1)_{|\mathbb{R}}\big)\big)$ such that

$$\frac{\partial v_i^\varepsilon}{\partial y_2} \twoheadrightarrow \xi_i^0 = \frac{\partial v_i^0}{\partial y_2} + \frac{\partial v_i^1}{\partial \eta_2},$$

and the second part of (4.3) holds.

Now, let $\varphi^\varepsilon(z) = \varphi\left(z_1, \dfrac{z_2}{\varepsilon h^\varepsilon(z_1)}, \dfrac{z_1}{\varepsilon}\right)$ for all $(z_1,z_2) \in \Omega^\varepsilon$. Recalling that $\mathrm{div}_z v^\varepsilon = 0$ in Ω^ε, we get

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega^\varepsilon} \left(\frac{\partial v_1^\varepsilon}{\partial z_1}(t,z) + \frac{\partial v_2^\varepsilon}{\partial z_2}(t,z)\right)\varphi^\varepsilon(z)\theta(t)\,dzdt \\
&= -\int_0^T \int_{\Omega^\varepsilon} \left(v_1^\varepsilon(t,z)\frac{\partial \varphi^\varepsilon}{\partial z_1}(z) + v_2^\varepsilon(t,z)\frac{\partial \varphi^\varepsilon}{\partial z_2}(z)\right)\theta(t)\,dzdt \\
&= -\int_0^T \int_{\Omega} \left(v_1^\varepsilon(t,y)\big(b_\varepsilon \cdot \nabla\varphi^\varepsilon\big)(y) + v_2^\varepsilon(t,y)\frac{1}{\varepsilon h^\varepsilon(y_1)}\frac{\partial \varphi^\varepsilon}{\partial y_2}(y)\right)\varepsilon h^\varepsilon(y_1)\theta(t)\,dydt \\
&= -\int_0^T \int_{\Omega} v_1^\varepsilon(t,y)\left(\varepsilon h\left(y_1, \frac{y_1}{\varepsilon}\right)\frac{\partial \varphi}{\partial y_1}\left(y, \frac{y}{\varepsilon}\right) + h\left(y_1,\frac{y_1}{\varepsilon}\right)\frac{\partial \varphi}{\partial \eta_1}\left(y,\frac{y}{\varepsilon}\right)\right. \\
&\quad \left. -y_2\left(\varepsilon\frac{\partial h}{\partial y_1}\left(y_1,\frac{y_1}{\varepsilon}\right) + \frac{\partial h}{\partial \eta_1}\left(y_1,\frac{y_1}{\varepsilon}\right)\right)\frac{\partial \varphi}{\partial y_2}\left(y,\frac{y}{\varepsilon}\right)\right)\theta(t)\,dydt \\
&\quad -\int_0^T \int_{\Omega} v_2^\varepsilon(t,y)\frac{\partial \varphi}{\partial y_2}\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt.
\end{aligned}$$

We pass to the limit as ε tends to zero:

$$\begin{aligned}
0 &= -\int_0^T \int_{\Omega\times(0,1)} v_1^0(t,y,\eta_1)\left(h(y_1,\eta_1)\frac{\partial \varphi}{\partial \eta_1}(y,\eta_1) - y_2\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\frac{\partial \varphi}{\partial y_2}(y,\eta_1)\right)\theta(t)\,d\eta_1 dydt \\
&\quad -\int_0^T \int_{\Omega\times(0,1)} v_2^0(t,y,\eta_1)\frac{\partial \varphi}{\partial y_2}(y,\eta_1)\theta(t)\,d\eta_1 dydt \\
&= \int_0^T \int_{\Omega\times(0,1)} \left(h(y_1,\eta_1)\frac{\partial v_1^0}{\partial \eta_1}(t,y,\eta_1) - y_2\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\frac{\partial v_1^0}{\partial y_2}(t,y,\eta_1) + \frac{\partial v_2^0}{\partial y_2}(t,y,\eta_1)\right)\varphi(y,\eta_1)\theta(t)\,d\eta_1 dydt
\end{aligned}$$

which gives (4.5). But, taking into account the boundary conditions on v^ε, we may reproduce the same computation with $\varphi \in \mathcal{C}^\infty\big(\overline{\Omega}; \mathcal{C}^\infty_\sharp(0,1)\big)$ such that φ is L-periodic in y_1, so with (4.5) it remains

$$\begin{aligned}
&\int_0^T \int_{\Gamma_1\times(0,1)} \left(-v_1^0(t,y,\eta_1)\frac{\partial h}{\partial \eta_1}(y_1,\eta_1) + v_2^0(t,y,\eta_1)\right)\varphi(y,\eta_1)\theta(t)\,d\eta_1 dy_1 dt \\
&\quad -\int_0^T \int_{\Gamma_0\times(0,1)} v_2^0(t,y,\eta_1)\varphi(y,\eta_1)\theta(t)\,d\eta_1 dy_1 dt = 0.
\end{aligned}$$

We choose more precisely $\varphi(y_1,y_2,\eta_1) = \hat{\varphi}(y_2)\tilde{\varphi}(y_1,\eta_1)$ with $\hat{\varphi} \in \mathcal{C}^\infty\big([0,1]\big)$ and $\tilde{\varphi} \in \mathcal{C}^\infty_\sharp\big([0,L]; \mathcal{C}^\infty_\sharp(0,1)\big)$. With $\hat{\varphi}(1) = 0$ and $\hat{\varphi}(0) = 1$ we get first

$$v_2^0 = 0 \quad \text{on } (0,T) \times \Gamma_0 \times (0,1).$$

Next with $\hat{\varphi}(1) = 1$ and $\hat{\varphi}(0) = 0$ we get

$$-v_1^0 \frac{\partial h}{\partial \eta_1} + v_2^0 = 0 \quad \text{on } (0,T) \times \Gamma_1 \times (0,1).$$

Finally, let $\varphi \in \mathcal{D}\big(0,L;\mathcal{C}^\infty_\sharp(0,1)\big)$ and $\varphi^\varepsilon(y_1,y_2) = \varphi\left(y_1,\dfrac{y_1}{\varepsilon}\right)(1-y_2)$ for all $(y_1,y_2)\in\Omega$. Taking into account the boundary conditions for v_1^ε (see (2.12)-(2.14)) we have

$$\int_0^T\int_\Omega \frac{\partial v_1^\varepsilon}{\partial y_2}(t,y)\varphi^\varepsilon(y)\theta(t)\,dydt = \int_0^T\int_\Omega v_1^\varepsilon(t,y)\varphi\left(y_1,\frac{y_1}{\varepsilon}\right)\theta(t)\,dydt.$$

By passing to the limit as ε tends to zero we obtain

$$\int_0^T\int_{\Omega\times(0,1)} \frac{\partial v_1^0}{\partial y_2}(t,y,\eta_1)\varphi(y_1,\eta_1)(1-y_2)\theta(t)\,d\eta_1 dydt$$
$$= \int_0^T\int_{\Omega\times(0,1)} v_1^0(t,y,\eta_1)\varphi(y_1,\eta_1)\theta(t)\,d\eta_1 dydt.$$

It follows that

$$\int_0^T\int_{\Gamma_0\times(0,1)} v_1^0(t,y,\eta_1)\varphi(y_1,\eta_1)\theta(t)\,d\eta_1 dydt = 0$$

which implies that $v_1^0 = 0$ on $(0,T)\times\Gamma_0\times(0,1)$. □

Similarly we can define the two-scale limit of Z^ε.

Proposition 4.4. **(Two-scale limit of the micro-rotation field)** *Under the assumptions of Proposition 3.2, there exist* $Z^0 \in L^2\big(0,T;L^2(\Omega;H^1_\sharp(Y))\big)$ *such that* $\dfrac{\partial Z^0}{\partial y_2} \in L^2\big(0,T;L^2(\Omega\times Y)\big)$ *and* $Z^1 \in L^2\big(0,T;L^2\big(\Omega\times(0,1);H^1_\sharp(0,1)_{/\mathbb{R}}\big)\big)$ *such that, possibly extracting a subsequence still denoted* $(Z^\varepsilon)_{\varepsilon>0}$*, we have*

$$Z^\varepsilon \twoheadrightarrow Z^0,\quad \frac{\partial Z^\varepsilon}{\partial y_2} \twoheadrightarrow \frac{\partial Z^0}{\partial y_2}+\frac{\partial Z^1}{\partial \eta_2}, \tag{4.12}$$

and

$$\varepsilon\frac{\partial Z^\varepsilon}{\partial y_1} \twoheadrightarrow \frac{\partial Z^0}{\partial \eta_1}. \tag{4.13}$$

Furthermore Z^0 *does not depend on* η_2*, and* $Z^0 \equiv 0$ *on* $(\Gamma_0\cup\Gamma_1)\times(0,1)\times(0,T)$.

Proof. The first part of the proof is identical to the proof of the previous proposition. Let us establish now the boundary conditions for the limit Z^0. Let $\theta\in\mathcal{D}(0,T)$, $\varphi\in\mathcal{C}^\infty\big(\overline{\Omega};\mathcal{C}^\infty_\sharp(0,1)\big)$ such that φ is L-periodic in y_1 and we define $\varphi^\varepsilon(z) = \varphi\left(z_1,\dfrac{z_2}{\varepsilon h^\varepsilon(z_1)},\dfrac{z_1}{\varepsilon}\right)$ for all $(z_1,z_2)\in\Omega^\varepsilon$. With the boundary conditions (2.12)-(2.14) for Z^ε we get

$$\int_0^T\int_{\Omega^\varepsilon}\frac{\partial Z^\varepsilon}{\partial z_2}(t,z)\varphi^\varepsilon(z)\theta(t)\,dzdt = -\int_0^T\int_{\Omega^\varepsilon} Z^\varepsilon(t,z)\frac{\partial\varphi^\varepsilon}{\partial z_2}(z)\theta(t)\,dzdt$$
$$= -\int_0^T\int_\Omega Z^\varepsilon(t,y)\frac{\partial\varphi}{\partial y_2}\left(y,\frac{y_1}{\varepsilon}\right)\theta(t)\,dydt = \int_0^T\int_\Omega \frac{\partial Z^\varepsilon}{\partial y_2}(t,y)\varphi\left(y,\frac{y_1}{\varepsilon}\right)\theta(t)\,dydt$$

and taking $\varepsilon\to 0^+$ we obtain

$$\int_0^T\int_{\Omega\times Y}\left(\frac{\partial Z^0}{\partial y_2}(t,y,\eta_1)+\frac{\partial Z^1}{\partial\eta_2}(t,y,\eta)\right)\varphi(y,\eta_1)\theta(t)\,d\eta_1 dydt$$
$$= -\int_0^T\int_{\Omega\times(0,1)} Z^0(t,y,\eta_1)\frac{\partial\varphi}{\partial y_2}(y,\eta_1)\theta(t)\,d\eta_1 dydt.$$

But the periodicity properties of Z^1 with respect to η_2 imply that

$$\int_0^T \int_{\Omega\times Y} \frac{\partial Z^1}{\partial \eta_2}(t,y,\eta)\varphi(y,\eta_1)\theta(t)\, d\eta_1 dy dt = 0.$$

Hence

$$\int_0^T \int_{\Omega\times(0,1)} \frac{\partial Z^0}{\partial y_2}(t,y,\eta_1)\varphi(y,\eta_1)\theta(t)\, d\eta_1 dy dt = -\int_0^T \int_{\Omega\times(0,1)} Z^0(t,y,\eta_1)\frac{\partial \varphi}{\partial y_2}(y,\eta_1)\theta(t)\, d\eta_1 dy dt.$$

By Green's formula we infer that

$$0 = -\int_0^T \int_{\Gamma_0\times(0,1)} Z^0(t,y,\eta_1)\varphi(y,\eta_1)\theta(t)\, d\eta_1 dy dt + \int_0^T \int_{\Gamma_1\times(0,1)} Z^0(t,y,\eta_1)\varphi(y,\eta_1)\theta(t)\, d\eta_1 dy dt.$$

Now we choose $\varphi(y_1,y_2,\eta_1) = \hat{\varphi}(y_2)\tilde{\varphi}(y_1,\eta_1)$ with $\hat{\varphi} \in \mathcal{C}^\infty\big([0,1]\big)$ and $\tilde{\varphi} \in \mathcal{C}^\infty_\sharp\big([0,L];\mathcal{C}^\infty_\sharp(0,1)\big)$, with $\hat{\varphi}(1)=0$, $\hat{\varphi}(0)=1$ then $\hat{\varphi}(1)=1$, $\hat{\varphi}(0)=0$, we get

$$0 = \int_0^T \int_{\Gamma_0\times(0,1)} Z^0(t,y,\eta_1)\tilde{\varphi}(y_1,\eta_1)\theta(t)\, d\eta_1 dy dt = \int_0^T \int_{\Gamma_1\times(0,1)} Z^0(t,y,\eta_1)\tilde{\varphi}(y_1,\eta_1)\theta(t)\, d\eta_1 dy dt$$

which allows us to conclude the proof of Proposition 4.4. ▯

Finally we can define the two-scale limit of p^ε.

Proposition 4.5. **(Two-scale limit of the pressure field)** *Under the assumptions of Proposition 3.3, there exists $p^0 \in H^{-1}\big(0,T;L^2(\Omega\times Y)\big)$ such that, possibly extracting a subsequence still denoted $(p^\varepsilon)_{\varepsilon>0}$, we have*

$$\varepsilon^2 p^\varepsilon \twoheadrightarrow p^0.$$

Moreover p^0 depends only t and y_1, $p^0 \in H^{-1}\big(0,T;H^1_\sharp(0,1)\big)$ and satisfies $\int_0^L p^0(t,y_1)\left(\int_0^1 h(y_1,\eta_1)\, d\eta_1\right) dy_1 = 0$ *almost everywhere in $(0,T)$.*

Proof. The first part of the result is an immediate consequence of the estimate (3.42) (see Proposition 3.3). From proposition 3.2 and (3.43) we know that there exists a constant $C>0$, independent of ε, such that for all $\varphi^\varepsilon \in H^1_0\big(0,T;H^1_0(\Omega)\big)$ we have

$$\left|\int_0^T \int_\Omega p^\varepsilon(t,y)\frac{\partial \varphi^\varepsilon}{\partial y_2}(t,y)\, dy dt\right| \le C\left(\|\varphi^\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \varepsilon\left\|\frac{\partial \varphi^\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}\right)$$
$$+\frac{C}{\varepsilon}\left(\|\varphi^\varepsilon\|_{L^\infty(0,T;L^4(\Omega))} + \|\varepsilon b_\varepsilon\cdot\nabla\varphi^\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial \varphi^\varepsilon}{\partial y_2}\right\|_{L^2(0,T;L^2(\Omega))}\right).$$

Now let $\varphi \in \mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(Y)\big)$ and $\theta \in \mathcal{D}(0,T)$. We define $\varphi^\varepsilon(t,y) = \theta(t)\varphi\left(y,\frac{y}{\varepsilon}\right)$ for all $(t,y)\in(0,T)\times\Omega$ and we get

$$\left|\int_0^T \int_\Omega \varepsilon^2 p^\varepsilon(t,y)\left(\frac{\partial \varphi}{\partial y_2}\left(t,y,\frac{y}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial \eta_2}\left(t,y,\frac{y}{\varepsilon}\right)\right)\theta(t)\, dy dt\right|$$
$$\le \mathcal{O}(\varepsilon) + C\|\theta\|_{C^0([0,T])}\left\|\frac{\partial \varphi}{\partial \eta_2}\right\|_{C^0(\overline{\Omega};C_\sharp(Y))}. \tag{4.14}$$

We multiply the two members of this inequality by ε and we pass to the limit as ε tends to zero. We obtain

$$\int_0^T \int_{\Omega\times Y} p^0(t,y,\eta)\frac{\partial\varphi}{\partial\eta_2}(y,\eta)\theta(t)\,d\eta dydt = 0.$$

Hence p^0 does not depend on η_2. Now we consider $\varphi \in \mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(0,1)\big)$ (i.e φ is independent of η_2) and we pass to the limit in (4.14) as ε tends to zero. We get

$$\int_0^T \int_{\Omega\times(0,1)} p^0(t,y,\eta)\frac{\partial\varphi}{\partial y_2}(y,\eta_1)\theta(t)\,d\eta dydt = 0$$

which implies that p^0 does not depend on y_2.

Now we take $\Theta = (\varphi^\varepsilon, 0, 0)$ in (3.14) and we multiply by ε: we get

$$\begin{aligned}
\varepsilon^2\int_0^T\int_\Omega p^\varepsilon\left(\frac{\partial\varphi^\varepsilon}{\partial y_1} - \frac{y_2}{h^\varepsilon}\frac{\partial h^\varepsilon}{\partial y_1}\frac{\partial\varphi^\varepsilon}{\partial y_2}\right)h^\varepsilon dydt &= -\varepsilon^2\int_0^T\int_\Omega v_1^\varepsilon\frac{\partial\varphi^\varepsilon}{\partial t}h^\varepsilon dydt\\
&+(\nu+\nu_r)\int_0^T\int_\Omega\left(h^\varepsilon(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon)(\varepsilon b_\varepsilon\cdot\nabla\varphi^\varepsilon) + \frac{1}{h^\varepsilon}\frac{\partial v_1^\varepsilon}{\partial y_2}\frac{\partial\varphi^\varepsilon}{\partial y_2}\right)dydt\\
+\int_0^T\int_\Omega\left(\varepsilon v_1^\varepsilon(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon)\varphi^\varepsilon h^\varepsilon + \varepsilon v_2^\varepsilon\frac{\partial v_1^\varepsilon}{\partial y_2}\varphi^\varepsilon\right)dydt &+ (\nu+\nu_r)\int_0^T\int_\Omega\frac{1}{h^\varepsilon}\frac{\partial U_\varepsilon}{\partial y_2}\frac{\partial\varphi^\varepsilon}{\partial y_2}dydt\\
&-2\nu_r\int_0^T\int_\Omega\varepsilon\frac{\partial Z^\varepsilon}{\partial y_2}\varphi^\varepsilon dydt + \int_0^T\int_\Omega U_\varepsilon(\varepsilon b_\varepsilon\cdot\nabla v_1^\varepsilon)\varphi^\varepsilon\varepsilon h^\varepsilon dydt\\
-\int_0^T\int_\Omega\varepsilon v_1^\varepsilon(\varepsilon b_\varepsilon\cdot\nabla\varphi^\varepsilon)U_\varepsilon h^\varepsilon dydt - \int_0^T\int_\Omega\varepsilon v_2^\varepsilon\frac{\partial\varphi^\varepsilon}{\partial y_2}U_\varepsilon dydt &- 2\varepsilon\nu_r\int_0^T\int_\Omega\frac{\partial W_\varepsilon}{\partial y_2}\varphi^\varepsilon dydt\\
&-\int_0^T\int_\Omega\left(f_1^\varepsilon\varphi^\varepsilon - \frac{\partial U^\varepsilon}{\partial t}\varphi^\varepsilon\right)\varepsilon^2 h^\varepsilon dydt,
\end{aligned}\tag{4.15}$$

where we recall that $U_\varepsilon(t,y) = U_0(t)\mathcal{U}(y_2h^\varepsilon(y_1))$ and $W_\varepsilon(t,y) = W_0(t)\mathcal{W}(y_2h^\varepsilon(y_1))$ for all $(t,y_1,y_2)\in[0,T]\times\Omega$. With the results of Proposition 3.2, we infer that there exists a constant $C>0$, independent of ε, such that

$$\begin{aligned}
&\left|\int_0^T\int_\Omega\varepsilon^2p^\varepsilon(t,y)(b_\varepsilon\cdot\nabla\varphi^\varepsilon)(t,y)h^\varepsilon(y)\,dydt\right|\\
&\le C\left(\|\varphi^\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|\varepsilon b_\varepsilon\cdot\nabla\varphi^\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial\varphi^\varepsilon}{\partial y_2}\right\|_{L^2(0,T;L^2(\Omega))} + \|\varphi^\varepsilon\|_{L^\infty(0,T;L^4(\Omega))}\right)\\
&+C\varepsilon^2\left\|\frac{\partial\varphi^\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

We multiply the two members of this inequality by ε and we obtain

$$\begin{aligned}
&\left|\int_0^T\int_\Omega\varepsilon^2p^\varepsilon(t,y)\left(\varepsilon\frac{\partial\varphi}{\partial y_1}\left(y_1,y_2,\frac{y_1}{\varepsilon}\right) + \frac{\partial\varphi}{\partial\eta_1}\left(y_1,y_2,\frac{y_1}{\varepsilon}\right)\right)h\left(y_1,\frac{y_1}{\varepsilon}\right)\theta(t)\,dydt\right.\\
&\left.-\int_0^T\int_\Omega\varepsilon^2p^\varepsilon(t,y)y_2\left(\varepsilon\frac{\partial h}{\partial y_1}\left(y_1,\frac{y_1}{\varepsilon}\right) + \frac{\partial h}{\partial\eta_1}\left(y_1,\frac{y_1}{\varepsilon}\right)\right)\frac{\partial\varphi}{\partial y_2}\left(y_1,y_2,\frac{y_1}{\varepsilon}\right)\theta(t)\,dydt\right| \le \mathcal{O}(\varepsilon)
\end{aligned}$$

By taking the limit as ε tends to zero, we have

$$\int_0^T \int_{\Omega\times(0,1)} p^0(t,y_1,\eta_1)\left(\frac{\partial\varphi}{\partial\eta_1}(y_1,y_2,\eta_1)h(y_1,\eta_1) - y_2\frac{\partial h}{\partial\eta_1}(y_1,\eta_1)\frac{\partial\varphi}{\partial y_2}(y_1,y_2,\eta_1)\right)\theta(t)\,d\eta_1 dydt = 0.$$

Reminding that p^0 is independent of y_2 and $\varphi\in\mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(0,1)\big)$, we get

$$\int_0^T \int_{\Omega\times(0,1)} p^0(t,y_1,\eta_1)\left(\frac{\partial\varphi}{\partial\eta_1}(y_1,\eta_1)h(y_1,y_2,\eta_1) - y_2\frac{\partial h}{\partial\eta_1}(y_1,\eta_1)\frac{\partial\varphi}{\partial y_2}(y_1,y_2,\eta_1)\right)\theta(t)\,d\eta_1 dydt$$

$$= \int_0^T \int_{\Omega\times(0,1)} p^0(t,y_1,\eta_1)\left(\frac{\partial\varphi}{\partial\eta_1}(y_1,y_2,\eta_1)h(y_1,\eta_1) + \frac{\partial h}{\partial\eta_1}(y_1,\eta_1)\varphi(y_1,y_2,\eta_1)\right)\theta(t)\,d\eta_1 dydt$$

$$= \int_0^T \int_{\Omega\times(0,1)} p^0(t,y_1,\eta_1)\frac{\partial(h\varphi)}{\partial\eta_1}(y_1,y_2,\eta_1)\,d\eta_1 dydt = 0.$$

Then for any $\phi\in\mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(0,1)\big)$, we may define $\varphi = \dfrac{\phi}{h}\in\mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(0,1)\big)$ and we obtain

$$\int_0^T \int_{\Omega\times(0,1)} p^0(t,y_1,\eta_1)\frac{\partial\phi}{\partial\eta_1}(y_1,y_2,\eta_1)\,d\eta_1 dydt = 0.$$

Thus we can conclude that p^0 is independent of η_1.

Now let $\varphi\in\mathcal{C}^\infty_\sharp(0,L)$ and $\theta\in\mathcal{D}(0,T)$. We define φ^ε by

$$\varphi^\varepsilon(y) = \frac{\varphi(y_1)}{h\left(y_1,\frac{y_1}{\varepsilon}\right)}\left(y_2e_1 + \varepsilon y_2^2\left(\frac{\partial h}{\partial y_1}\left(y_1,\frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial h}{\partial\eta_1}\left(y_1,\frac{y_1}{\varepsilon}\right)\right)e_2\right) \tag{4.16}$$

for all $(y_1,y_2)\in\Omega$. We can check that $\varphi^\varepsilon\in\tilde{V}$ and with Lemma 3.1, Proposition 3.2 and (3.43)-(4.15), we obtain

$$\left|\int_0^T \int_\Omega \varepsilon^2 p^\varepsilon\left((b_\varepsilon\cdot\nabla\varphi_1^\varepsilon) + \frac{1}{\varepsilon h^\varepsilon}\frac{\partial\varphi_2^\varepsilon}{\partial y_2}\right)h^\varepsilon\theta(t)\,dydt\right| \le \mathcal{O}(\varepsilon) + C\|\varphi\theta\|_{L^2((0,T)\times(0,L))}$$

with a constant $C>0$ independent of ε. Hence

$$\left|\int_0^T \int_\Omega \varepsilon^2 p^\varepsilon(t,y)y_2\frac{\partial\varphi}{\partial y_1}(y_1)\theta(t)\,dydt\right| \le \mathcal{O}(\varepsilon) + C\|\varphi\theta\|_{L^2((0,T)\times(0,L))}.$$

We pass to the limit as ε tends to zero:

$$\left|\int_0^T \int_\Omega p^0(t,y_1)y_2\frac{\partial\varphi}{\partial y_1}(y_1)\theta(t)\,dydt\right| = \frac{1}{2}\left|\int_0^T \int_0^L p^0(t,y_1)\frac{\partial\varphi}{\partial y_1}(y_1)\theta(t)\,dydt\right| \le C\|\varphi\theta\|_{L^2((0,T)\times(0,L))}$$

and we infer that $p^0\in H^{-1}\big(0,T;H^1_\sharp(0,L)\big)$.

Finally, recalling that $\displaystyle\int_{\Omega^\varepsilon} p^\varepsilon(t,z)\,dz = 0$ almost everywhere in $(0,T)$, we have

$$\int_0^T \int_\Omega \varepsilon^2 p^\varepsilon(t,y)h^\varepsilon(y)\theta(t)\,dydt = 0 \quad \forall\theta\in\mathcal{D}(0,T)$$

and by passing to the limit as ε tends to zero, we get

$$\int_0^T \int_{\Omega\times(0,1)} p^0(t,y_1,\eta_1)h(y_1,\eta_1)\theta(t)\,d\eta_1 dydt = 0 \quad \forall\theta\in\mathcal{D}(0,T)$$

which allows us to conclude the proof of Proposition 4.5. ☐

5. The limit problem. Now let us pass to the limit in equation (2.17). It is convenient to introduce the following functional spaces:

$$\tilde{V} = \Big\{\varphi \in \big(\mathcal{C}^\infty(\overline{\Omega};\mathcal{C}^\infty_\sharp(0,1))\big)^2;\ \varphi \text{ is L-periodic in } y_1,\ \varphi = 0 \text{ on } \Gamma_0\times(0,1),$$
$$-\varphi_1\frac{\partial h}{\partial \eta_1} + \varphi_2 = 0 \text{ on } \Gamma_1\times(0,1)\Big\}$$

$$\tilde{V}_{div} = \left\{\varphi \in \tilde{V};\ h\frac{\partial \varphi_1}{\partial \eta_1} - y_2\frac{\partial h}{\partial \eta_1}\frac{\partial \varphi_1}{\partial y_2} + \frac{\partial \varphi_2}{\partial y_2} = 0 \text{ in } \Omega\times(0,1)\right\}$$

$$\tilde{H}^1 = \big\{\psi \in \mathcal{C}^\infty(\overline{\Omega};\mathcal{C}^\infty_\sharp(0,1));\ \psi \text{ is L-periodic in } y_1,\ \psi = 0 \ \text{ on } (\Gamma_0\cup\Gamma_1\times(0,1)\big\},$$

$$V_{div} = \text{ closure of } \tilde{V}_{div} \text{ in } L^2_\sharp\big(0,L;\mathcal{F}\big)^2,$$
$$H^1_{0,\sharp} = \text{closure of } \tilde{H}^1 \text{ in } L^2_\sharp\big(0,L;\mathcal{F}\big)$$

with

$$\mathcal{F} = \left\{v \in L^2\big((0,1);H^1_\sharp(0,1)\big),\ \frac{\partial v}{\partial y_2} \in L^2\big((0,1)\times(0,1)\big)\right\}.$$

THEOREM 5.1. *Assume that there exist* $f \in \mathcal{C}\big([0,T];\mathcal{C}\big(\overline{\Omega};\mathcal{C}_\sharp(0,1)\big)\big)^2$ *and* $g \in \mathcal{C}\big([0,T];\mathcal{C}\big(\overline{\Omega};\mathcal{C}_\sharp(0,1)\big)\big)$ *such that* f *and* g *are L-periodic in* y_1 *and*

$$\varepsilon^2 f^\varepsilon(t,y) = f\left(t,y,\frac{y_1}{\varepsilon}\right),\quad \varepsilon^2 g^\varepsilon(t,y) = g\left(t,y,\frac{y_1}{\varepsilon}\right)\quad \forall (t,y)\in[0,T]\times\overline{\Omega}.$$

Then the functions v^0, Z^0 *and* p^0 *satisfy the following limit problem:*

$$\begin{aligned}
&(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}\sum_{i=1}^2\left(h(\bar{b}\cdot\nabla v_i^0)(\bar{b}\cdot\nabla\varphi_i) + \frac{1}{h}\frac{\partial v_i^0}{\partial y_2}\frac{\partial \varphi_i}{\partial y_2}\right)\theta\, d\eta_1 dy dt\\
&+\alpha\int_0^T\int_{\Omega\times(0,1)}\left(h(\bar{b}\cdot\nabla Z^0)(\bar{b}\cdot\nabla\psi) + \frac{1}{h}\frac{\partial Z^0}{\partial y_2}\frac{\partial \psi}{\partial y_2}\right)\theta\, d\eta_1 dy dt\\
&-\int_0^T\int_{\Omega\times(0,1)}\frac{\partial p^0}{\partial y_1}h\varphi_1\theta\, d\eta_1 dy dt\\
&= -(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}\left(h(\bar{b}\cdot\nabla \bar{U})(\bar{b}\cdot\nabla\varphi_1) + \frac{1}{h}\frac{\partial \bar{U}}{\partial y_2}\frac{\partial \varphi_1}{\partial y_2}\right)\theta\, d\eta_1 dy dt\\
&-\alpha\int_0^T\int_{\Omega\times(0,1)}\left(h(\bar{b}\cdot\nabla \bar{W})(\bar{b}\cdot\nabla\psi) + \frac{1}{h}\frac{\partial \bar{W}}{\partial y_2}\frac{\partial \psi}{\partial y_2}\right)\theta\, d\eta_1 dy dt\\
&+\int_0^T\int_{\Omega\times(0,1)} f\varphi h\theta\, d\eta_1 dy dt + \int_0^T\int_{\Omega\times(0,1)} g\psi h\theta\, d\eta_1 dy dt
\end{aligned}$$

for all $\Theta = (\varphi,\psi) \in V_{div}\times H^1_{0\sharp}$ *and* $\theta \in \mathcal{D}(0,T)$*, where* $\bar{b}\cdot\nabla$ *is the differential operator defined by*

$$\bar{b}\cdot\nabla = \left(1, -\frac{y_2}{h(y_1,\eta_1)}\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\right)\begin{pmatrix}\dfrac{\partial}{\partial \eta_1}\\[2mm] \dfrac{\partial}{\partial y_2}\end{pmatrix}$$

and

$$\bar{U}(t, y_1, y_2, \eta_1) = U_0(t)\mathcal{U}\left(h(y_1, \eta_1)y_2\right),$$
$$\bar{W}(t, y_1, y_2, \eta_1) = W_0(t)\mathcal{W}\left(h(y_1, \eta_1)y_2\right)$$

for all $(t, y_1, y_2, \eta_1, t) \in [0, T] \times \overline{\Omega} \times [0, 1]$.

Proof. With the above assumptions for f^ε and g^ε we can check immediately that

$$\|\varepsilon^2 f^\varepsilon\|_{L^2((0,T)\times\Omega)} \le \sqrt{T|\Omega|}\|f\|_{\mathcal{C}\left([0,T];\mathcal{C}\left(\overline{\Omega};\mathcal{C}_\sharp(0,1)\right)\right)}, \quad \|\varepsilon^2 g^\varepsilon\|_{L^2((0,T)\times\Omega)} \le \sqrt{T|\Omega|}\|g\|_{\mathcal{C}\left([0,T];\mathcal{C}\left(\overline{\Omega};\mathcal{C}_\sharp(0,1)\right)\right)}$$

and

$$\varepsilon^2 f^\varepsilon \twoheadrightarrow f, \quad \varepsilon^2 g^\varepsilon \twoheadrightarrow g.$$

Let us recall that

$$b_\varepsilon \cdot \nabla = \left(1, -\frac{y_2}{h^\varepsilon(y_1)}\frac{\partial h^\varepsilon}{\partial y_1}(y_1)\right)\begin{pmatrix} \dfrac{\partial}{\partial y_1} \\ \\ \dfrac{\partial}{\partial y_2} \end{pmatrix}$$

Taking into account the convergence results of Proposition 4.3 and Proposition 4.4, we get

$$\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon = \varepsilon\frac{\partial v_i^\varepsilon}{\partial y_1}(y) - \frac{y_2}{h\left(y_1, \frac{y_1}{\varepsilon}\right)}\left(\varepsilon\frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right) + \frac{\partial h}{\partial \eta_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\right)\frac{\partial v_i^\varepsilon}{\partial y_2}(y)$$
$$\twoheadrightarrow \frac{\partial v_i^0}{\partial \eta_1}(y, \eta_1) - \frac{y_2}{h(y_1, \eta_1)}\frac{\partial h}{\partial \eta_1}(y_1, \eta_1)\left(\frac{\partial v_i^0}{\partial y_2}(y, \eta_1) + \frac{\partial v_i^1}{\partial \eta_2}(y, \eta)\right) = \bar{b} \cdot \nabla v_i^0 - \frac{y_2}{h}\frac{\partial h}{\partial \eta_1}\frac{\partial v_i^1}{\partial \eta_2}$$

for $i = 1, 2$ and

$$b_\varepsilon \cdot \nabla Z^\varepsilon \twoheadrightarrow \bar{b} \cdot \nabla Z^0 - \frac{y_2}{h}\frac{\partial h}{\partial \eta_1}\frac{\partial Z^1}{\partial \eta_2}.$$

Similarly, let $\phi \in \mathcal{C}^\infty\left(\overline{\Omega}; \mathcal{C}_\sharp^\infty(0,1)\right)$ and $\phi^\varepsilon(y_1, y_2) = \phi\left(y_1, y_2, \dfrac{y_1}{\varepsilon}\right)$ for all $(y_1, y_2) \in \Omega$. We have

$$b_\varepsilon \cdot \nabla \phi^\varepsilon = \frac{\partial \phi^\varepsilon}{\partial y_1}(y) - \frac{y_2}{h\left(y_1, \frac{y_1}{\varepsilon}\right)}\left(\frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\right)\frac{\partial \phi^\varepsilon}{\partial y_2}(y)$$
$$= \frac{\partial \phi}{\partial y_1}\left(y, \frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial \phi}{\partial \eta_1}\left(y, \frac{y_1}{\varepsilon}\right) - \frac{y_2}{h\left(y_1, \frac{y_1}{\varepsilon}\right)}\left(\frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\right)\frac{\partial \phi}{\partial y_2}\left(y, \frac{y_1}{\varepsilon}\right).$$

Now let $\theta \in \mathcal{D}(0, T)$, $\Theta = (\varphi, \psi) \in \tilde{V}_{div} \times \tilde{H}^1$ and let $\Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon)$ with

$$\varphi^\varepsilon(z) = \varphi\left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon}\right) + \frac{z_2}{h^\varepsilon(z_1)}\frac{\partial h}{\partial y_1}\left(z_1, \frac{z_1}{\varepsilon}\right)\varphi_1\left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon}\right)e_2$$

and $\psi^\varepsilon(z) = \psi\left(z_1, \dfrac{z_2}{\varepsilon h^\varepsilon(z_1)}, \dfrac{z_1}{\varepsilon}\right)$ for all $(z_1, z_2) \in \Omega^\varepsilon$. We have $\Theta^\varepsilon \in \tilde{V}^\varepsilon \times \tilde{H}^{1,\varepsilon}$ and from (3.4)

$$\varepsilon\int_0^T a\left(\bar{v}^\varepsilon(t), \Theta^\varepsilon\right)\theta(t)\,dt \to (\nu + \nu_r)\int_0^T \int_{\Omega\times Y}\sum_{i=1}^2\left(h(\bar{b}\cdot\nabla v_i^0)(\bar{b}\cdot\nabla\varphi_i) + \frac{1}{h}\frac{\partial v_i^0}{\partial y_2}\frac{\partial \varphi_i}{\partial y_2}\right)\theta\,d\eta dy dt$$

$$+\alpha \int_0^T \int_{\Omega\times Y} \left(h(\bar{b}\cdot\nabla Z^0)(\bar{b}\cdot\nabla\psi) + \frac{1}{h}\frac{\partial Z^0}{\partial y_2}\frac{\partial \psi}{\partial y_2}\right)\theta\, d\eta dy dt$$
$$+(\nu+\nu_r)\int_0^T \int_{\Omega\times Y} \sum_{i=1}^2 \left(\left(-y_2 \frac{\partial h}{\partial \eta_1}\frac{\partial v_i^1}{\partial \eta_2}\right)(\bar{b}\cdot\nabla\varphi_i) + \frac{1}{h}\frac{\partial v_i^1}{\partial \eta_2}\frac{\partial \varphi_i}{\partial y_2}\right)\theta\, d\eta dy dt$$
$$+\alpha \int_0^T \int_{\Omega\times Y} \left(\left(-y_2 \frac{\partial h}{\partial \eta_1}\frac{\partial Z^1}{\partial \eta_2}\right)(\bar{b}\cdot\nabla\psi) + \frac{1}{h}\frac{\partial Z^1}{\partial \eta_2}\frac{\partial \psi}{\partial y_2}\right)\theta\, d\eta dy dt.$$

But these last two integral terms vanish since φ, ψ and h do not depend on η_2 and v^1 and Z^1 are η_2-periodic. Hence we obtain

$$\varepsilon \int_0^T a\big(\bar{v}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt \to \int_0^T \bar{a}\big(\bar{v}^0(t), \Theta\big)\theta(t)\, dt$$

with $\bar{v}^0 = (v^0, Z^0)$ and

$$\bar{a}(\bar{v}, \Theta) = (\nu+\nu_r)\int_{\Omega\times(0,1)} \sum_{i=1}^2 \left(h(\bar{b}\cdot\nabla v_i)(\bar{b}\cdot\nabla\varphi_i) + \frac{1}{h}\frac{\partial v_i}{\partial y_2}\frac{\partial \varphi_i}{\partial y_2}\right) d\eta_1 dy$$
$$+\alpha \int_{\Omega\times(0,1)} \left(h(\bar{b}\cdot\nabla Z)(\bar{b}\cdot\nabla\psi) + \frac{1}{h}\frac{\partial Z}{\partial y_2}\frac{\partial \psi}{\partial y_2}\right) d\eta_1 dy$$

for all $\bar{v} = (v, Z) \in V_{div} \times H^1_{0\sharp}$, for all $\Theta = (\varphi, \psi) \in V_{div} \times H^1_{0\sharp}$.

From (3.5) and the estimates (3.15)-(3.16)-(3.18) obtained at Proposition 3.2 we get

$$\varepsilon \int_0^T B\big(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt = \mathcal{O}(\varepsilon) \to 0$$

and similarly, from (3.6) and (3.15)-(3.16)-(3.18)

$$\varepsilon \int_0^T \mathcal{R}\big(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt = \mathcal{O}(\varepsilon) \to 0.$$

Let us consider now the right hand side of equation (2.18). We recall that $\bar{\xi}^\varepsilon = (U^\varepsilon e_1, W^\varepsilon)$ with

$$U^\varepsilon(t,z) = U_0(t)\mathcal{U}\left(h^\varepsilon(y_1)y_2\right) = \bar{U}\left(t, y_1, y_2, \frac{y_1}{\varepsilon}\right)$$
$$W^\varepsilon(t,z) = W_0(t)\mathcal{W}\left(h^\varepsilon(y_1)y_2\right) = \bar{W}\left(t, y_1, y_2, \frac{y_1}{\varepsilon}\right)$$

and U_0, W_0 belong to $H^1(0,T)$, $\mathcal{U}$, $\mathcal{W}$ belong to $\mathcal{D}\big((-\infty, h_{max})\big)$. Hence $\bar{U}$ and $\bar{W}$ belong to $\mathcal{C}\big([0,T]; \mathcal{C}^1(\overline{\Omega}; \mathcal{C}^1_\sharp(0,1))\big)$ and with (3.9)-(3.10)-(3.11)-(3.12)

$$\varepsilon \int_0^T a\big(\bar{\xi}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt \to \int_0^T \bar{a}\big(\bar{\xi}(t), \Theta\big)\theta(t)\, dt$$
$$\varepsilon \int_0^T B\big(\bar{\xi}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt = \mathcal{O}(\varepsilon) \to 0$$
$$\varepsilon \int_0^T B\big(\bar{v}^\varepsilon(t), \bar{\xi}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt = \mathcal{O}(\varepsilon) \to 0$$
$$\varepsilon \int_0^T \mathcal{R}\big(\bar{\xi}^\varepsilon(t), \Theta^\varepsilon\big)\theta(t)\, dt = \mathcal{O}(\varepsilon) \to 0$$

with $\bar{\xi} = (\bar{U}e_1, \bar{W})$.

Next, using (3.13) and reminding that $\varphi^\varepsilon \in \tilde{V}^\varepsilon$:

$$
\begin{aligned}
&\varepsilon \int_0^T \int_{\Omega^\varepsilon} p^\varepsilon(t,z)\mathrm{div}_z \varphi^\varepsilon(z)\theta(t)\, dzdt = \int_0^T \int_\Omega \varepsilon p^\varepsilon \left((\varepsilon b_\varepsilon \cdot \nabla \varphi_1^\varepsilon) + \frac{1}{h^\varepsilon}\frac{\partial \varphi_2^\varepsilon}{\partial y_2}\right) h^\varepsilon \theta \, dydt \\
&= \int_0^T \int_\Omega \varepsilon p^\varepsilon \left(\varepsilon h\left(y_1, \frac{y_1}{\varepsilon}\right) \frac{\partial \varphi_1}{\partial y_1}\left(y, \frac{y}{\varepsilon}\right) + h\left(y_1, \frac{y_1}{\varepsilon}\right)\frac{\partial \varphi_1}{\partial \eta_1}\left(y, \frac{y}{\varepsilon}\right)\right. \\
&-y_2 \left(\varepsilon \frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right) + \frac{\partial h}{\partial \eta_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\right) \frac{\partial \varphi_1}{\partial y_2}\left(y, \frac{y}{\varepsilon}\right) + \frac{\partial \varphi_2}{\partial y_2}\left(y, \frac{y}{\varepsilon}\right) \\
&\left. + \varepsilon y_2 \frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\frac{\partial \varphi_1}{\partial y_2}\left(y, \frac{y}{\varepsilon}\right) + \varepsilon \frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\varphi_1\left(y, \frac{y}{\varepsilon}\right)\right)\theta(t)\, dydt \\
&= \int_0^T \int_\Omega \varepsilon^2 p^\varepsilon \left(h\left(y_1, \frac{y_1}{\varepsilon}\right)\frac{\partial \varphi_1}{\partial y_1}\left(y, \frac{y}{\varepsilon}\right) + \frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\varphi_1\left(y, \frac{y}{\varepsilon}\right)\right)\theta(t)\, dydt \\
&\to \int_0^T \int_{\Omega\times(0,1)} p^0 \frac{\partial (h\varphi_1)}{\partial y_1}\theta \, d\eta_1 dydt = -\int_0^T \int_{\Omega\times(0,1)} \frac{\partial p^0}{\partial y_1} h\varphi_1 \theta\, d\eta_1 dydt.
\end{aligned}
$$

Finally

$$
\begin{aligned}
&\varepsilon^2 \int_0^T \frac{d}{dt}\big[\bar{v}^\varepsilon, \Theta^\varepsilon\big](t)\theta(t)\, dt = -\varepsilon^2 \int_0^T \big[\bar{v}^\varepsilon, \Theta^\varepsilon\big](t)\theta'(t)\, dt = \mathcal{O}(\varepsilon^2) \to 0 \\
&-\varepsilon^2 \int_0^T \left[\frac{\partial \bar{\xi}^\varepsilon}{\partial t}, \Theta^\varepsilon\right](t)\theta(t)\, dt = \mathcal{O}(\varepsilon^2) \to 0.
\end{aligned}
$$

By multiplying equation (2.18) by $\varepsilon\theta(t)$, integrating over $[0,T]$ and passing to the limit as ε tends to zero we obtain

$$
\begin{aligned}
&\int_0^T \bar{a}\big(\bar{v}^0(t), \Theta\big)\theta(t)\, dt - \int_0^T \int_{\Omega\times(0,1)} \frac{\partial p^0}{\partial y_1} h\varphi_1\theta\, d\eta_1 dydt \\
&= -\int_0^T \bar{a}\big(\bar{\xi}(t), \Theta\big)\theta(t)\, dt + \int_0^T \int_{\Omega\times(0,1)} (f\varphi + g\psi)h\theta\, d\eta_1 dydt
\end{aligned}
$$

for all $\Theta = (\varphi, \psi) \in \tilde{V}_{div} \times \tilde{H}^1$ and $\theta \in \mathcal{D}(0,T)$. By density of $\tilde{V}_{div} \times \tilde{H}^1$ into $V_{div} \times H^1_{0\sharp}$ we get the announced result. □

We may observe that the limit problem is totally decoupled with respect to the velocity and micro-rotation fields. Furthermore the time variable appears as a parameter in the limit problem. More precisely, for all $y_1 \in [0,L]$, let a_{y_1} be the bilinear symmetric form defined on $\mathcal{F}$ by

$$
a_{y_1}(w,\psi) = \int_Y \left(h(y_1,\eta_1)(\bar{b}\cdot\nabla w)(y_2,\eta_1)(\bar{b}\cdot\nabla\psi)(y_2,\eta_1) + \frac{1}{h(y_1,\eta_1)}\frac{\partial w}{\partial y_2}(y_2,\eta_1)\frac{\partial \psi}{\partial y_2}(y_2,\eta_1)\right) d\eta_1 dy_2
$$

for all $(w,\psi) \in \mathcal{F}$. The limit velocity, pressure and micro-rotation fields are solution of the problems (P_{v^0,p^0}) and (P_{Z^0}) given respectively by

Find $v^0 \in L^2(0,T;V_{div})$ and $p^0 \in H^{-1}(0,T;H^1_\sharp(0,L))$ such that
$\displaystyle\int_0^L p^0(t,y_1)\left(\int_0^1 h(y_1,\eta_1)\, d\eta_1\right) dy_1 = 0$ a.e. $t \in [0,T]$ and

$$(\nu+\nu_r)\int_0^L \sum_{i=1}^2 a_{y_1}(v_i^0,\varphi_i)\,dy_1 - \int_0^L \frac{\partial p^0}{\partial y_1}\left(\int_0^1 h(y_1,\cdot)\varphi_1\,d\eta_1\right)\,dy_1$$
$$= -(\nu+\nu_r)\int_0^L a_{y_1}\big(\bar{U}(t),\varphi_1\big)\,dy_1 + \int_0^L \left(\int_Y f(t,y_1,\cdot,\cdot)h(y_1,\cdot)\varphi\,d\eta_1 dy_2\right)dy_1$$
$$\forall \varphi \in V_{div},\ \text{a.e. } t\in[0,T]$$

and

Find $Z^0 \in L^2(0,T;H^1_{0,\sharp})$ such that
$$\alpha\int_0^L a_{y_1}(Z^0,\psi)\,dy_1 = -\alpha\int_0^L a_{y_1}\big(\bar{W}(t),\psi\big)\,dy_1 + \int_0^L \left(\int_Y g(t,y_1,\cdot,\cdot)h(y_1,\cdot)\psi\,d\eta_1 dy_2\right)dy_1$$
$$\forall \psi \in H^1_{0,\sharp},\ \text{a.e. } t\in[0,T].$$

Proposition 5.2. *Under the assumptions of theorem 5.1, the limit microrotation field Z^0 is uniquely given by*

$$Z^0(t,y_1,y_2,\eta_1) = W_0(t)z^1_{y_1}(y_2,\eta_1) + z^2_{t,y_1}(y_2,\eta_1) \quad \text{a.e. in } (0,T)\times\Omega\times(0,1)$$

where $z^1_{y_1}\in H^1_{0,\sharp}$ and $z^2_{t,y_1}\in H^1_{0,\sharp}$ are the unique solutions of the following auxiliary problems:

$$a_{y_1}(z^1_{y_1},\psi) = -a_{y_1}\big(\mathcal{W}(y_1,\cdot),\psi\big) \quad \forall\psi\in H^1_{0,\sharp}$$

and

$$\alpha a_{y_1}(z^2_{t,y_1},\psi) = \int_Y g_{t,y_1} h(y_1,\cdot)\psi\,d\eta_1 dy_2 \quad \forall\psi\in H^1_{0,\sharp}.$$

Proof. It is clear that, for all $y_1\in[0,L]$, the mapping a_{y_1} is continuous on $\mathcal{F}$. Moreover

$$a_{y_1}(w,w) \geq h_{min}\|\bar{b}\cdot\nabla w\|^2_{L^2(Y)} + \frac{1}{h_{max}}\left\|\frac{\partial w}{\partial z_2}\right\|_{L^2(Y)}$$

and

$$\|\bar{b}\cdot\nabla w\|^2_{L^2(Y)} = \left\|\frac{\partial w}{\partial \eta_1}\right\|^2_{L^2(Y)} + \int_Y \frac{y_2^2}{h^2(y_1,\eta_1)}\left(\frac{\partial h}{\partial \eta_1}(y_1,\eta_1,t)\right)^2\left(\frac{\partial w}{\partial y_2}\right)^2 d\eta_1 dy_2$$
$$-2\int_Y \frac{y_2}{h(y_1,\eta_1)}\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\frac{\partial w}{\partial y_2}\frac{\partial w}{\partial \eta_1}\,d\eta_1 dy_2$$
$$\geq (1-\lambda)\left\|\frac{\partial w}{\partial \eta_1}\right\|^2_{L^2(Y)} + \left(1-\frac{1}{\lambda}\right)\int_Y \frac{y_2^2}{h^2(y_1,\eta_1)}\left(\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\right)^2\left(\frac{\partial w}{\partial y_2}\right)^2 d\eta_1 dy_2 \quad \forall\lambda>0.$$

But, recalling that $h\in\mathcal{C}\big([0,L]\times[0,1]\big)$, there exists $C>0$, independent of y_1, such that

$$\left|\frac{y_2}{h(y_1,\eta_1)}\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\right| \leq C \quad \forall (y_1,y_2,\eta_1)\in[0,L]\times Y$$

and, for all $\lambda \in (0,1)$

$$a_{y_1}(w,w) \geq C_1(\lambda) \left\| \frac{\partial w}{\partial \eta_1} \right\|^2_{L^2(Y)} + C_2(\lambda) \left\| \frac{\partial w}{\partial y_2} \right\|^2_{L^2(Y)}, \tag{5.1}$$

where $C_1(\lambda) = h_{min}(1-\lambda)$ and $C_2(\lambda) = \Big(\left(1 - \frac{1}{\lambda}\right) C^2 h_{min} + \frac{1}{h_{max}} \Big)$. Then we may choose λ such that

$$\lambda \in \left(\frac{C^2 h_{max} h_{min}}{1 + C^2 h_{max} h_{min}}, 1 \right) \tag{5.2}$$

which shows that a_{y_1} is coercive on $H^1_{0,\sharp}$, uniformly with respect to y_1. Since $g \in \mathcal{C}\big([0,T]; \mathcal{C}\big(\overline{\Omega}; \mathcal{C}_\sharp(0,1)\big)\big)$ the mapping $g_{t,y_1} = g(t, y_1, \cdot, \cdot)$ belongs to $L^2(Y)$ for all $(t, y_1) \in [0,T] \times [0,L]$. Then Lax-Milgram's theorem implies that, for all $(t,y_1) \in [0,T] \times [0,L]$ the problems

$$\begin{aligned}
&\text{Find } z^1_{y_1} \in H^1_{0,\sharp} \text{ such that} \\
&a_{y_1}(z^1_{y_1}, \psi) = -a_{y_1}\big(\mathcal{W}(y_1,\cdot), \psi\big) \quad \forall \psi \in H^1_{0,\sharp}
\end{aligned}$$

and

$$\begin{aligned}
&\text{Find } z^2_{t,y_1} \in H^1_{0,\sharp} \text{ such that} \\
&\alpha a_{y_1}(z^2_{t,y_1}, \psi) = \int_Y g_{t,y_1} h(y_1,\cdot) \psi \, d\eta_1 dy_2 \quad \forall \psi \in H^1_{0,\sharp}
\end{aligned}$$

admit a unique solution. Furthermore, recalling that $W_0 \in H^1(0,T) \subset \mathcal{C}\big([0,T]\big)$ and $h \in \mathcal{C}^1\big([0,L] \times [0,1]; \mathbb{R}\big)$ with values in $[h_{min}, h_{max}] \subset \mathbb{R}^+_*$, we infer that the mapping $(t, y_1) \mapsto Z^0_{t,y_1} = W_0 z^1_{y_1} + z^2_{t,y_1}$ is continuous on $[0,T] \times [0,L]$ with values in $H^1_{0,\sharp}$ and is L-periodic in y_1.

Thus the mapping $Z^0 : (t, y_1, y_2, \eta_1) \mapsto Z^0_{t,y_1}(y_2, \eta_1)$ belongs to $L^2(0,T; H^1_{0,\sharp})$ and solves the problem (P_{Z_0}). Indeed, let $\psi \in \tilde{H}^1$. Then $\psi(y_1, \cdot, \cdot) \in H^1_\sharp$ and we get

$$\begin{aligned}
&\alpha a_{y_1}\big(Z^0_{t,y_1}, \psi(y_1,\cdot,\cdot)\big) = -\alpha a_{y_1}\big(\bar{W}(t, y_1, \cdot, \cdot), \psi(y_1, \cdot, \cdot)\big) \\
&+ \int_Y g(t, y_1, \cdot, \cdot) h(y_1, \cdot) \psi \, d\eta_1 dy_2 \quad \forall y_1 \in [0,L].
\end{aligned}$$

Both sides of this equality are continuous on $[0,L]$, hence we may integrate with respect to y_1 and

$$\int_0^L a_{y_1}(Z^0_{t,y_1}, \psi) \, dy_1 = -\int_0^L a_{y_1}\big(\bar{W}, \psi\big) \, dy_1 + \int_0^L \left(\int_Y g_{t,y_1} h(y_1,\cdot) \psi \, d\eta_1 dy_2 \right) dy_1 \quad \forall \psi \in \tilde{H}^1.$$

It follows that

$$\begin{aligned}
&\int_0^L a_{y_1}(Z^0(t), \psi) \, dy_1 = -\int_0^L a_{y_1}\big(\bar{W}(t), \psi\big) \, dy_1 + \int_0^L \left(\int_Y g(t) h(y_1,\cdot) \psi \, d\eta_1 dy_2 \right) dy_1 \\
&\forall \psi \in \tilde{H}^1, \text{ a.e. } t \in [0,T]
\end{aligned}$$

and the density of $\tilde{H}^1$ into $H^1_{0,\sharp}$ allows us to conclude the existence part of the proof. Then we observe that the uniqueness is a immediate consequence of the uniform coercivity of a_{y_1} with respect to y_1. $\square$

Now, for all $y_1 \in [0, L]$, let

$$\tilde{V}_{y_1} = \left\{ \varphi \in \big(\mathcal{C}^\infty\big([0,1];\mathcal{C}^\infty_\sharp(0,1)\big)\big)^2;\ \varphi(0,\cdot) = 0 \text{ on } (0,1),\ -\varphi_1(1,\cdot)\frac{\partial h}{\partial y_1}(y_1,\cdot) + \varphi_2(1,\cdot) = 0 \text{ on } (0,1) \right\},$$

$$\tilde{V}_{y_1,div} = \left\{ \varphi \in \tilde{V}_{y_1};\ h(y_1,\cdot)\frac{\partial \varphi_1}{\partial \eta_1} - y_2 \frac{\partial h}{\partial \eta_1}(y_1,\cdot)\frac{\partial \varphi_1}{\partial y_2} + \frac{\partial \varphi_2}{\partial y_2} = 0 \text{ in } Y \right\}$$

and

$$V_{y_1,div} = \text{closure of } \tilde{V}_{y_1,div} \text{ in } \mathcal{F}^2.$$

Let $\bar{a}_{y_1}(w,\varphi) = (\nu + \nu_r)\sum_{i=1}^{2} a_{y_1}(w_i,\varphi_i)$ for all $(w,\varphi) \in V^2_{y_1,div}$. With Poincaré's inequality we know that $w \mapsto \|\nabla w\|_{L^2(Y)}$ defines a norm on $V_{y_1,div}$ which is equivalent to the H^1-norm. Furthermore, with (5.1)-(5.2), we may infer that $\bar{a}_{y_1}$ is coercive on $V_{y_1,div}$ for all $y_1 \in [0,L]$, uniformly with respect to y_1. It follows that we can define $w^1_{y_1} \in V_{y_1,div}$, $w^2_{y_1} \in V_{y_1,div}$ and $w^3_{t,y_1} \in V_{y_1,div}$ as the unique solutions of

$$\bar{a}_{y_1}(w^1_{y_1},\varphi) = -\int_Y h(y_1,\cdot)\varphi_1\, d\eta_1 dy_2 \quad \forall \varphi \in V_{y_1,div},$$

$$\bar{a}_{y_1}(w^2_{y_1},\varphi) = -(\nu+\nu_r)a_{y_1}\big(\mathcal{U}(y_1,\cdot),\varphi_1\big) \quad \forall \varphi \in V_{y_1,div}$$

and

$$\bar{a}_{y_1}(w^3_{t,y_1},\varphi) = \int_Y f_{t,y_1} h(y_1,\cdot)\varphi_1\, d\eta_1 dy_2 \quad \forall \varphi \in V_{y_1,div}$$

with $f_{t,y_1} = f(t,y_1,\cdot,\cdot)$ for all $(t,y_1) \in [0,T]\times[0,L]$.

Then we have

Proposition 5.3. *Under the assumptions of theorem 5.1, the limit velocity v^0 is uniquely given by*

$$v^0(t,y_1,y_2,\eta_1) = \frac{\partial p^0}{\partial y_1}(t,y_1)w^1_{y_1}(y_2,\eta_1) + U_0(t)w^2_{y_1}(y_2,\eta_1) + w^3_{t,y_1}(y_2,\eta_1) \quad \text{a.e. in } (0,T)\times\Omega\times(0,1).$$

Furthermore, for almost every $t \in [0,T]$, the limit pressure $p^0(t,\cdot)$ is the unique solution in $H^1_\sharp(0,L)_{|\mathbb{R}}$ of the following homogenized Reynolds equation

$$\int_0^L \frac{\partial p^0}{\partial y_1}\frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1},w^1_{y_1})\,dy_1 = -\int_0^L U_0(t)\frac{\partial \psi}{\partial y_1}\bar{a}\big(w^1_{y_1},w^2_{y_1}\big)\,dy_1 - \int_0^L \frac{\partial \psi}{\partial y_1}\bar{a}\big(w^1_{y_1},w^3_{t,y_1}\big)\,dy_1 \quad \forall \psi \in H^1_\sharp(0,L)$$

satisfying $\int_0^L p^0\left(\int_0^1 h(\cdot,\eta_1)\,d\eta_1\right)dy_1 = 0.$

Proof. The first part of the result is obtained by using the same kind of arguments as in Proposition 5.2.

Let $\theta \in \mathcal{D}(0,T)$, $\psi \in \mathcal{C}^\infty_\sharp\big([0,L]\big)$ and $\psi^\varepsilon(z) = \psi(z_1)$ for all $z = (z_1, z_2) \in \Omega^\varepsilon$. Recalling that $\mathrm{div}_z v^\varepsilon = 0$ in Ω^ε and using the boundary conditions (2.12)-(2.13)-(2.14) we get

$$0 = \frac{1}{\varepsilon}\int_0^T \int_{\Omega^\varepsilon} \left(\frac{\partial v_1^\varepsilon}{\partial z_1}(t,z) + \frac{\partial v_2^\varepsilon}{\partial z_2}(t,z) \right) \psi^\varepsilon(z)\theta(t)\, dzdt$$

$$0 = -\frac{1}{\varepsilon}\int_0^T \int_{\Omega^\varepsilon} v_1^\varepsilon(t,z)\frac{\partial \psi^\varepsilon}{\partial z_1}(z)\theta(t)\, dzdt = -\int_0^T \int_\Omega v_1^\varepsilon(t,y)(b_\varepsilon \cdot \nabla \psi^\varepsilon)(y) h^\varepsilon(y)\theta(t)\, dydt$$

$$= -\int_0^T \int_\Omega v_1^\varepsilon(t,y)\frac{\partial \psi}{\partial y_1}(y_1) h\left(y_1, \frac{y_1}{\varepsilon}\right)\theta(t)\, dydt.$$

By passing to the limit as ε tends to zero we get

$$0 = \int_0^T \int_{\Omega\times(0,1)} v_1^0(t,y,\eta_1)\frac{\partial \psi}{\partial y_1}(y_1) h(y_1,\eta_1)\theta(t)\, d\eta_1 dydt.$$

It follows that

$$\int_0^L \frac{\partial p^0}{\partial y_1}\frac{\partial \psi}{\partial y_1}\left(\int_Y w^1_{y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2\right) dy_1 + \int_0^L U_0(t)\frac{\partial \psi}{\partial y_1}\left(\int_Y w^2_{y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2\right) dy_1$$
$$+ \int_0^L \frac{\partial \psi}{\partial y_1}\left(\int_Y w^3_{t,y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2\right) dy_1 = 0 \quad \text{a.e. } t \in [0,T].$$

But

$$\int_Y w^1_{y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2 = -\bar{a}_{y_1}(w^1_{y_1}, w^1_{y_1}),$$
$$\int_Y w^2_{y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2 = -\bar{a}_{y_1}(w^1_{y_1}, w^2_{y_1}),$$
$$\int_Y w^3_{t,y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2 = -\bar{a}_{y_1}(w^1_{y_1}, w^3_{y_1})$$

and by density of $\mathcal{C}^\infty_\sharp\big([0,L]\big)$ in $H^1_\sharp(0,L)$ we get

$$\int_0^L \frac{\partial p^0}{\partial y_1}\frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1}, w^1_{y_1})\, dy_1 = -\int_0^L U_0(t)\frac{\partial \psi}{\partial y_1}\bar{a}\big(w^1_{y_1}, w^2_{y_1}\big)\, dy_1$$
$$-\int_0^L \frac{\partial \psi}{\partial y_1}\bar{a}\big(w^1_{y_1}, w^3_{t,y_1}\big)\, dy_1 \quad \forall \psi \in H^1_\sharp(0,L), \text{ a.e. } t \in [0,T].$$

We can check that this Reynolds problem admits a unique solution in $H^1_\sharp(0,L)_{|\mathbb{R}}$. Indeed, let $\varphi_{y_1}(y_2,\eta_1) = \left(\dfrac{-y_2 + y_2^2}{h(y_1,\eta_1)}, \dfrac{\partial h}{\partial \eta_1}(y_1,\eta_1)\dfrac{y_2^2(y_2-1)}{h(y_1,\eta_1)} \right)$ for all $(y_2,\eta_1) \in Y$, for all $y_1 \in [0,L]$. Then we obtain $\varphi_{y_1} \in V_{y_1,div}$ and

$$\bar{a}_{y_1}(w^1_{y_1}, \varphi_{y_1}) = -\int_Y h(y_1,\eta_1)\varphi_{y_1,1}(y_2,\eta_1)\, d\eta_1 dy_2 = \frac{1}{6}.$$

Since $\bar{a}_{y_1}$ defines an inner product on $V_{y_1,div}$, we have

$$\frac{1}{6} = \bar{a}_{y_1}(w^1_{y_1}, \varphi_{y_1}) \le \bar{a}_{y_1}(w^1_{y_1}, w^1_{y_1})^{1/2}\bar{a}_{y_1}(\varphi_{y_1}, \varphi_{y_1})^{1/2}.$$

But the mapping $y_1 \mapsto \bar{a}_{y_1}(\varphi_{y_1}, \varphi_{y_1})$ is continuous on $[0, L]$ and does not vanish since $\varphi_{y_1} \not\equiv 0$. It follows that there exists $\alpha > 0$ such that $\bar{a}_{y_1}(\varphi_{y_1}, \varphi_{y_1}) \geq \alpha$ for all $y_1 \in [0, L]$ and $\bar{a}(w^1_{y_1}, w^1_{y_1}) \geq \frac{1}{36\alpha}$ for all $y_1 \in [0, L]$.

We may observe also that the mapping $\psi \mapsto -\int_0^L U_0(t)\frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1}, w^2_{y_1})\, dy_1 - \int_0^L \frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1}, w^3_{t,y_1})\, dy_1$ is linear and continuous on $H^1_\sharp(0, L)$ for every $t \in [0, T]$ and the mapping $(p, \psi) \mapsto \int_0^L \frac{\partial p}{\partial y_1}\frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1}, w^1_{y_1})\, dy_1$ is bilinear, symmetric, continuous and coercive on $H^1_\sharp(0, L)_{|\mathbb{R}}$. We can apply Lax-Milgram's theorem to conclude the proof of Proposition 5.3. □

As a consequence of the uniqueness of p^0, we can state the next result:

THEOREM 5.4. *The whole sequences $(\varepsilon^2 p^\varepsilon)_{\varepsilon>0}$, $(v^\varepsilon)_{\varepsilon>0}$ and $(Z^\varepsilon)_{\varepsilon>0}$ satisfy the following convergence:*

$$\begin{aligned} &\varepsilon p^\varepsilon \twoheadrightarrow p^0 \\ &v^\varepsilon \twoheadrightarrow v^0 \\ &Z^\varepsilon \twoheadrightarrow Z^0. \end{aligned}$$

6. Concluding remarks. A possible generalization of this study consists in considering a domain Ω^ε where both the upper and lower boundary are oscillating. More precisely, let us assume that

$$\Omega^\varepsilon = \big\{(z_1, z_2) \in \mathbb{R}^2; \quad 0 < z_1 < L, \quad -\varepsilon\beta(z_1)h^\varepsilon(z_1) < z_2 < \varepsilon h^\varepsilon(z_1)\big\}$$

where β belongs to $\mathcal{C}^\infty([0, L]; \mathbb{R}^+)$ and is L-periodic in z_1 (with $\beta \equiv 0$ we recognize the case presented in the previous sections). Now we should denote by Γ^ε_0 the lower boundary of Ω^ε and we can choose the functions $\mathcal{U}$ and $\mathcal{W}$ (see Lemma 2.1) such that $\mathcal{U}$ and $\mathcal{W}$ belong to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with $\mathcal{U}(\sigma) = \mathcal{W}(\sigma) = 1$ for all $\sigma \leq 0$ and $\mathrm{Supp}(\mathcal{U}) \subset (-\infty, h_m)$, $\mathrm{Supp}(\mathcal{W}) \subset (-\infty, h_m)$. Then we define again

$$U^\varepsilon(t, z_2) = \mathcal{U}^\varepsilon(z_2)U_0(t) = \mathcal{U}(\frac{z_2}{\varepsilon})U_0(t), \quad W^\varepsilon(t, z_2) = \mathcal{W}^\varepsilon(z_2)W_0(t) = \mathcal{W}(\frac{z_2}{\varepsilon})W_0(t)$$

and we get the same variational problem (P^ε). It follows that the existence and uniqueness result given at Theorem 2.2 is still valid. Furthermore, we can use the same scalings (see (3.1) and (3.2)) which transforms the domain Ω^ε into

$$\Omega = \big\{(y_1, y_2) \in \mathbb{R}^2; \quad 0 < y_1 < L, \quad -\beta(y_1) < y_2 < 1\big\}$$

and by reproducing the same computations, we obtain the same a priori estimates as in Proposition 3.2 and Proposition 3.3.

Finally we may apply once again the two-scale convergence technique to pass to the limit as ε tends to zero. We obtain the same convergence properties for the velocity and the micro-rotaion field as in Proposition 4.3 and Proposition 4.4 with $\Gamma_0 = \big\{\big(y_1, -\beta(y_1)\big);\ 0 < y_1 < L\big\}$. For the convergence of the pressure, we follow the same arguments as in Proposition 4.5 with a natural modification of the test-function φ^ε introduced at formula (4.16) which may be chosen now as

$$\varphi^\varepsilon(y) = \frac{\varphi(y_1)}{h\left(y_1, \frac{y_1}{\varepsilon}\right)}\left(\big(y_2 + \beta(y_1)\big)e_1 + \varepsilon y_2\big(y_2 + \beta(y_1)\big)\left(\frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\right)e_2\right)$$

for all $(y_1, y_2) \in \Omega$, which leads to

$$\left| \int_0^T \int_0^L p^0(t, y_1) \frac{\partial}{\partial y_1} \big(\frac{1}{2} (1+\beta)^2 \varphi \big)(y_1) \theta(t) \, dy dt \right| \leq C \|\varphi \theta\|_{L^2((0,T)\times(0,L))}.$$

Then we may conclude by considering any $\phi \in \mathcal{C}^\infty_\sharp(0, L)$ and letting $\varphi = \frac{2\phi}{(1+\beta)^2}$.

Hence the limit problem remains the same as in Theorem 5.1: Z^0 and v^0 can be decomposed by using the same auxiliary problems and p^0 is the unique solution of the same Reynolds equation, with obvious adaptations in the definition of a_{y_1} and $\bar{a}_{y_1}$, i.e. for all $y_1 \in [0, L]$:

$$\begin{aligned} a_{y_1}(w, \psi) = \int_{-\beta(y_1)}^1 \int_0^1 \Big(& h(y_1, \eta_1)(\bar{b} \cdot \nabla w)(y_2, \eta_1)(\bar{b} \cdot \nabla \psi)(y_2, \eta_1) \\ & + \frac{1}{h(y_1, \eta_1)} \frac{\partial w}{\partial y_2}(y_2, \eta_1) \frac{\partial \psi}{\partial y_2}(y_2, \eta_1) \Big) \, d\eta_1 dy_2 \end{aligned}$$

for all $(w, \psi) \in \mathcal{F}_{y_1} = \Big\{ v \in L^2\big((-\beta(y_1), 1); H^1_\sharp(0,1)\big); \dfrac{\partial v}{\partial y_2} \in L^2\big((-\beta(y_1), 1) \times (0,1)\big) \Big\}$ and $\bar{a}_{y_1}(w, \varphi) = (\nu + \nu_r) \sum_{i=1}^2 a_{y_1}(w_i, \varphi_i)$ for all $(w, \varphi) \in V^2_{y_1,div}$ where $V_{y_1,div}$ is the closure of $\tilde{V}_{y_1,div}$ in $\mathcal{F}^2_{y_1}$ and

$$\begin{aligned} \tilde{V}_{y_1,div} = \Big\{ & \varphi \in \big(\mathcal{C}^\infty\big([-\beta(y_1), 1]; \mathcal{C}^\infty_\sharp(0,1)\big)\big)^2; \ \varphi(-\beta(y_1), \cdot) = 0 \text{ on } (0,1), \\ & -\varphi_1(1, \cdot) \frac{\partial h}{\partial y_1}(y_1, \cdot) + \varphi_2(1, \cdot) = 0 \text{ on } (0,1), \\ & h(y_1, \cdot) \frac{\partial \varphi_1}{\partial \eta_1} - y_2 \frac{\partial h}{\partial \eta_1}(y_1, \cdot) \frac{\partial \varphi_1}{\partial y_2} + \frac{\partial \varphi_2}{\partial y_2} = 0 \text{ in } (-\beta(y_1), 1) \times (0,1) \Big\}. \end{aligned}$$

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